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***In Memoriam**

ALGEBRAS, GROUPS AND GEOMETRIES

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**ON CERTAIN CONSTRUCTIONS OF LIE (SUPER) ALGEBRAS
AND SYMMETRIC (SUPER) SPACES ASSOCIATED WITH
(ε, δ)-FREUDENTHAL-KANTOR TRIPLE SYSTEMS**

-In homage of Professor Susumu Okubo (1930-2015)

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Abstract

The main purpose of this paper is to study constructions of Lie algebras and symmetric spaces associated with (ε, δ) -Freudenthal-Kantor triple systems based on certain concept of ternary product (triple product) without using root systems in Lie algebras. Next, by means of the constructions, we will give several examples of bisymmetric (super)spaces.

Keywords: triple systems, symmetric spaces, Lie (super)algebras construction.

MSC classification: Primary 17A40 ternary compositions, 17B60 Lie (super)algebras associated with other structures, 17C50 Jordan structures associated with other structures. Secondary 53B25 Local submanifolds, 53C35 Differential geometry of symmetric spaces.

Introduction and Motivation

This note is to deal with certain constructions of Lie (super)algebras and of symmetric spaces based on our subjects called a triple system (or ternary algebra). We give several examples of triple systems (or ternary products) equipped with the balanced property, then by using this triple product's concept, we exhibit a construction of symmetric spaces (Lie triple systems) and simple Lie algebras without using root systems, in particular, our purpose is to consider a construction of bisymmetric spaces from balanced Freudenthal-Kantor triple systems. That is, it seems that this study in nonassociative algebra field with respect to Jordan and Lie structures is in close contact with a symmetric (super)space equipped with complex structure of differential geometry, since the tangent space of the symmetric (super)space is a δ -Lie triple system ($\delta = \pm 1$). Hereby we exhibit main our results described in Section 2 and Section 4 of this article.

Theorem A *For a balanced $(1,1)$ -Freudenthal-Kanmtot triple system U , the Lie algebra $L(U)$ and the Lie triple system $T(U)$ associated with U , we have subspaces of quaternionic symmetric spaces as follows;*

$$F_4/(C_3 \oplus A_1) < E_6/(A_5 \oplus A_1) < E_7/(D_6 \oplus A_1) < E_8/(E_7 \oplus A_1).$$

Theorem B *For a balanced $(-1,-1)$ -Freudenthal-Kantor triple system U , the Lie superalgebra $L(U)$ and the anti-Lie triple system $T(U)$ associated with U , we have subspaces of symmetric superspaces as follows;*

- $osp(2|2)/(A_1 \oplus gl(1)) < osp(1|4)/(A_1 \oplus A_1) < F(4)/(A_1 \oplus B_3),$
- $osp(1|2)/A_1 < osp(3|2)/(A_1 \oplus A_1) < G(3)/(A_1 \oplus G_2).$

From mathematical history's viewpoint, the concept discussed here first appeared with a class of nonassociative algebras, that is commutative Jordan algebras, which was the defining subspace g_{-1} in the Tits-Kantor-Koecher (for short TKK) construction of 3-graded Lie algebras $g = g_{-1} \oplus g_0 \oplus g_1$, such that $[g_i, g_j] \subseteq g_{i+j}$. Nonassociative algebras are rich in algebraic structures, and they provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry (for example, [12], [51], [64], [66], [69]). Specially, the concept of nonassociative algebras such as Jordan and Lie (super)algebras plays an important role in many

mathematical and physical subjects ([6], [11]-[14], [16], [23], [27], [29], [30], [33], [35], [36], [37], [50], [59], [60], [65], [70]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([1]-[4], [7]-[9], [21], [24], [25], [45]-[50], [55]-[58], [61]-[63]) by using the standard embedding method ([23], [53], [54], [62], [68]). In particular, the generalized Jordan triple system of second order, or $(-1, 1)$ -Freudenthal Kantor triple system (for short $(-1, 1)$ -FKTS), is a useful concept ([14]-[22], [46]-[49], [52], [67]) for the constructions of simple Lie algebras, while the $(-1, -1)$ -FKTS plays the same role ([7], [23], [26], [28], [31], [34], [35]) for the construction of Lie superalgebras, while the δ -Jordan Lie triple systems act similarly for that of Jordan superalgebras ([24], [25], [61]). Specially, we have constructed a model of basic Lie superalgebras $D(2, 1; \alpha)$, $G(3)$ and $F(4)$ ([23], [26], [28]).

For a simple(primary) example of our motivations, if we decompose the associative binary product of the matrix algebra $Mat(n, n; \Phi)$ in its symmetric and skew-symmetric parts

$$xy = \frac{1}{2}(xy + yx) + \frac{1}{2}(xy - yx),$$

then the second term leads to the Lie algebra $gl(n, \Phi)$ with the product $[x, y] = xy - yx$, also, by the triple product $[xyz] = [[x, y], z]$, it leads to the Lie triple system, and the first term leads to the Jordan algebra $Mat(n, n; \Phi)$ with the product $x \cdot y = \frac{1}{2}(xy + yx)$, also by the triple product $\{xyz\} = \frac{1}{2}(x^t yz + z^t yx)$, where $^t x$ denote the transpose matrix of x , it leads to the Jordan triple system.

As a final comment of this introduction, we provide well-known results due to O. Loos as follows; if A is a unital commutative Jordan algebra with a unital element e , that is, satisfying $(xy)x^2 = x(y(x^2))$, and $xy = yx$, then the triple product given by

$$\{xyz\} = (xy)z + x(yz) - y(xz)$$

defines a Jordan triple system with $\{xey\} = xy$, i.e., it satisfies the two relations $\{xy\{abc\}\} = \{\{xya\}bc\} - \{a\{yxb\}c\} + \{ab\{xyc\}\}$ (this relation is often called a fundamental identity), and $\{xyz\} = \{zyx\}$ (this relation is

called a commutative identity, since $xy = \{xey\} = \{yex\} = yx$ and next the new triple product $[xyz]$ given by

$$[xyz] = \{xyz\} - \{yxz\}$$

defines a Lie triple system.

Briefly summarizing this article, we will generalize these results and exhibit examples of Lie (super)algebras associated with Freudenthal-Kantor triple systems which is a generalization of the Jordan triple system. Toward to its applications, in particular, we will give a construction of symmetric (super)spaces with an almost complex structure (i.e., equipped with Nijenhuis operator). And we will provide an idea of quaternionic bisymmetric spaces associated with our constructions.

Roughly describing, we have an illustration for our concept ;

Algebraic structures $\iff$ Geometric structures.

For examples, it seems that there are certain algebraic structures associated with symmetric, R-symmetric, homogeneous spaces, totally geodesic manifolds, and symmetric domains, further Yang-Baxter equations etc. Hence, the notations of triple systems appeared in this article is utility with respect to the characterization of geometry or mathematical physics. In particular, it emphasizes that our objects are to give a representation of symmetric (super)spaces from concepts of triple systems (ternary algebras).

1 Definitions and Preliminaries

In this paper triple systems have finite dimension being defined over a field Φ of characteristic $\neq 2$ or 3 , unless otherwise specified. In order to render the paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order) ([46]-[50]) .

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \rightarrow V$, $(x, y, z) \mapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \quad (1)$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) - K(a, b)L(x, y) = 0, \quad (2)$$

where $L(a, b)c := (abc)$ and $K(a, b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the following condition

$$(abc) = (cba), \text{ i.e., } K(a, c)b = 0. \quad (3)$$

The JTS is a special case in the GJTS of 2nd order since $K(x, y) \equiv 0$.

We next can generalize the concept of GJTS of 2nd order as follows (see [14], [15], [19], [23], [29], [38] [68] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \quad (4)$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) + \varepsilon K(a, b)L(x, y) = 0, \quad (5)$$

where

$$L(a, b)c := (abc), \quad K(a, b)c := (acb) - \delta(bca), \quad (6)$$

is called an (ε, δ) -Freudenthal-Kantor triple system (for short (ε, δ) -FKTS).

An (ε, δ) -FKTS is said to be *unitary* if $Id \in \{K(a, b)\}_{span}$.

A triple system satisfying only the identity (4) is called a *generalized FKTS* (for short GFKTS), while the identity (5) is called the *second order condition* (this condition needs to construct of 5-graded Lie (super)algebras).

Remark From the relation Eq. (6), we note that

$$K(b, a) = -\delta K(a, b). \quad (7)$$

A triple system is called a (α, β, γ) *triple system associated with a bilinear form* if

$$(xyz) = \alpha \langle x, y \rangle z + \beta \langle y, z \rangle x + \gamma \langle z, x \rangle y,$$

where $\langle x, y \rangle$ is a bilinear form such that $\langle x, y \rangle = \kappa \langle y, x \rangle$, $\kappa = \pm 1$, $\alpha, \beta, \gamma \in \Phi$.

With respect to a classification of this triple system, we would like to refer ([42]).

An (ε, δ) -FKTS is said to be *balanced* if there is a bilinear form $\langle x, y \rangle \in \Phi^*$ such that $K(x, y) = \langle x, y \rangle Id$, that is, $\dim_{\Phi} \{K(x, y)\}_{span} = 1$ holds.

Remark We note that a balanced triple system (i.e., it fulfills $K(x, y) = \langle x, y \rangle Id$) is unitary, since $Id \in \{K(x, y)\}_{span}$.

Triple products are denoted by (xyz) , $\{xyz\}$, $[xyz]$ and $\langle xyz \rangle$ upon their suitability.

Remark We note that the concept of GJTS of 2nd order coincides with that of $(-1, 1)$ -FKTS. Thus we can construct the corresponding Lie algebras by means of the standard embedding method ([7], [14]-[20], [22], [23], [26], [28], [48]).

For $\delta = \pm 1$, a triple system $(a, b, c) \mapsto [abc], a, b, c \in V$ is called a δ -Lie triple system (for short δ -LTS) if the following three identities are fulfilled

$$\begin{aligned} [abc] &= -\delta[bac], \\ [abc] + [bca] + [cab] &= 0, \\ [ab[xyz]] &= [[abx]yz] + [x[aby]z] + [xy[abz]], \end{aligned} \quad (8)$$

where $a, b, x, y, z \in V$. An 1-LTS is a LTS while a -1 -LTS is an *anti-LTS*, by ([15]). Note that the set $L(V, V)$ of all left multiplications $L(x, y)$ of V is a Lie subalgebra of $Der V$, where we denote by $L(x, y)z = [xyz]$.

Theorem 1.1 ([14]-[17], [23]) *Let $(U(\varepsilon, \delta), \langle xyz \rangle)$ be an (ε, δ) -FKTS. If J is an endomorphism of $U(\varepsilon, \delta)$ such that $J \langle xyz \rangle = \langle JxJyJz \rangle$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon, \delta), [xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] := \langle xJyz \rangle - \delta \langle yJxz \rangle + \delta \langle xJzy \rangle - \langle yJzx \rangle. \quad (9)$$

Remark Note that for the case of $\varepsilon = -1, \delta = 1$ and $K(x, y) = 0$, we have a special case in Prop.1.1, that is, it implies that $J = Id$, $\{xyz\}$ is the JTS and $[xyz] = \{xyz\} - \{yxz\}$ is the LTS described in Introduction.

Corollary ([14]) *Let $U(\varepsilon, \delta)$ be an (ε, δ) -FKTS. Then the vector space $T(\varepsilon, \delta) = U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} L(a, d) - \delta L(c, b) & \delta K(a, c) \\ -\varepsilon K(b, d) & \varepsilon(L(d, a) - \delta L(b, c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}. \quad (10)$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie

superalgebra (if $\delta = -1$), $L(U(\varepsilon, \delta)) = D(T(\varepsilon, \delta), T(\varepsilon, \delta)) \oplus T(\varepsilon, \delta)$, associated with $T(\varepsilon, \delta)$ where $D(T(\varepsilon, \delta), T(\varepsilon, \delta))$ is the set of inner derivations of $T(\varepsilon, \delta)$;

$$D(T(\varepsilon, \delta), T(\varepsilon, \delta)) := \left\{ \begin{pmatrix} L(a, b) & \delta K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{pmatrix} \right\}_{\text{span}},$$

$$T(\varepsilon, \delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U(\varepsilon, \delta) \right\}_{\text{span}}.$$

We note that $L(X, Y)Z = [XYZ] = [[X, Y], Z]$, $\forall X, Y, Z \in T(\varepsilon, \delta)$.

From $K(x, y) \in \text{End } U(\varepsilon, \delta)$, we use the following notation:

$$\mathbf{k} := \{K(x, y) | x, y \in U(\varepsilon, \delta)\}, \{EFG\} := EFG + GFE, \forall E, F, G \in \mathbf{k}.$$

Then, we may make a structure of a JTS $\mathbf{k}$ with respect to the triple product $\{EFG\} \in \mathbf{k}$, hence $[EFG] = \{EFG\} - \{FEG\}$ has a structure of LTS ([21]).

We next introduce an analogue of Nijenhuis tensor in differential geometry defined by

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y], \forall X, Y \in T(\varepsilon, \delta)$$

and $J = \begin{pmatrix} 0 & \varepsilon \\ -\delta & 0 \end{pmatrix}$, that is $J^2 = -\varepsilon\delta I\mathbf{d}$, hence if $J^2 = -I\mathbf{d}$, then this (the case of $\varepsilon\delta = 1$) has a structure of almost complex on the δ LTS $T(\varepsilon, \delta)$.

Theorem 1.2 *Let $U = U(\varepsilon, \delta)$ be a (ε, δ) -FKTS, $T(U) = T(U(\varepsilon, \delta))$ be the δ -LTS and $L(U) = L(U(\varepsilon, \delta))$ be the standard embedding Lie (super)algebra associated with U . Then the following are equivalent:*

- (i) $N(X, Y) = 0, \forall X, Y \in T(\varepsilon, \delta)$,
- (ii) $\varepsilon\delta L(y, x) - \varepsilon L(x, y) = K(x, y), \forall x, y \in U(\varepsilon, \delta)$.

This $J \in \text{End } T(\varepsilon, \delta)$ may generalize on $\tilde{J} \in \text{End } L(U)$ defined by

$$\tilde{J} := JD(X, Y)J^{-1} \oplus JZ, \forall X, Y, Z \in T(\varepsilon, \delta).$$

Then we note that $\tilde{J}$ has an interesting property, for example, an automorphism of $L(U)$ associated with U .

Theorem 1.3 *For a (ε, δ) -FKTS U and $L(U)$ as in above Proposition, assuming $\varepsilon = \delta$ and $K(x, y) = L(y, x) - \varepsilon L(x, y)$, then the elements $f =$*

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in sl(2)$ (i.e., $[f, g] = h, [f, h] = -2f, [g, h] = 2g$) are derivations of $L(U)$. Furthermore, the Lie (super)algebra $L(U)$ has a structure of $sl(2)$ -module.

Remark We note that $L(U) = L(U(\varepsilon, \delta)) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the five graded Lie (super)algebra such that $U(\varepsilon, \delta) \oplus U(\varepsilon, \delta) = L_{-1} \oplus L_1 = T(\varepsilon, \delta)$ (δ -LTS), $L_{-2} = \mathbf{k}$ (JTS) and $D(T(\varepsilon, \delta), T(\varepsilon, \delta)) = L_{-2} \oplus L_0 \oplus L_2$ (the derivation of $T(\varepsilon, \delta)$) equipped with $[L_i, L_j] \subseteq L_{i+j}$ and $L_{-1} \oplus L_1 = L(U)/L_{-2} \oplus L_0 \oplus L_2$. In Introduction, we had used the notation $g = g_{-1} \oplus g_0 \oplus g_1$ instead of $L_{-1} \oplus L_0 \oplus L_1$. This Lie (super)algebra construction is one of reasons to study nonassociative algebras and triple systems without using root systems (for a Lie superalgebra, refer to ([10], [13], [65])). Also from another viewpoint, this construction can be represented by the concept of a normal triality algebra (see [34], [35]).

This section is a survey of our papers with respect to the triple systems and the construction of Lie (super)algebras mainly, if it need, the readers would like to see the earlier our references of therein.

Remark Note that if $K(x, y) = \langle x, y \rangle Id$ and $\delta = 1$, from Eq.(5), then we obtain $K(x, y) = L(y, x) - \varepsilon L(x, y)$, and by $\delta = 1$, we have $K(x, y) = -K(y, x)$. That is, balanced and $\delta = 1 \implies \varepsilon = 1$. Also, balanced and $\delta = -1 \implies \varepsilon = -1$. Furthermore, if $\langle x, y \rangle$ is nondegenerate, the standard embedding Lie algebra $L(U)$ is simple ([16], [19], [20], [32]).

Following ([17], [19], [20], [36], [37]), with respect a generalized vector matrix $X = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in M(V)$, where $\alpha, \beta \in \Phi$, $\mathbf{a}, \mathbf{b} \in V = H_3(\mathbf{O})$, where $\mathbf{O}$ is an octonion algebra over the field Φ , in order to make as self-contained as possible, we recall first the definition of an outer product $\times$ and a bilinea form $B(x, y)$ in the exceptional Jordan algebra $H_3(\mathbf{O})$ as follows;

$$A \times B := 2A \cdot B - (Tr A)B - (Tr B)A + [(Tr A)(Tr B) - Tr (A \cdot B)]E$$

and $B(A, B) := Tr (A \cdot B)$ for $A, B \in H_3(\mathbf{O})$, where $A \cdot B = \frac{1}{2}(AB + BA)$. Secondary we consider either the set of matrices $Mat(m, n; \Phi)$ with coefficients in Φ or the set of the generalized vector matrix;

$$M(V) = \left\{ \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \mid \alpha, \beta \in \Phi, \mathbf{a}, \mathbf{b} \in V = Mat(m, n; \Phi) \text{ or } H_3(\mathbf{O}) \right\}.$$

In this $M(V)$, we introduce a bilinear operator $\circ$

$$\begin{pmatrix} \alpha_1 & \mathbf{a}_1 \\ \mathbf{b}_1 & \beta_1 \end{pmatrix} \circ \begin{pmatrix} \alpha_2 & \mathbf{a}_2 \\ \mathbf{b}_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + B(\mathbf{a}_1, \mathbf{b}_2) & \alpha_1\mathbf{a}_2 + \beta_2\mathbf{a}_1 + \mathbf{b}_1 \times \mathbf{b}_2 \\ \alpha_2\mathbf{b}_1 + \beta_1\mathbf{b}_2 + \mathbf{a}_1 \times \mathbf{a}_2 & \beta_2\beta_1 + B(\mathbf{a}_2, \mathbf{b}_1) \end{pmatrix}. \quad (11)$$

Next we define the involutions

$$P : \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \longrightarrow \begin{pmatrix} -\alpha & \mathbf{a} \\ -\mathbf{b} & \beta \end{pmatrix} \quad \text{and} \quad - : \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \longrightarrow \begin{pmatrix} \beta & \mathbf{a} \\ \mathbf{b} & \alpha \end{pmatrix}.$$

Then by using above notations, we can obtain a triple product on $M(V)$ as follows:

$$(\spadesuit) \quad < x_1 x_2 x_3 > = x_1 \circ (\overline{Px_2} \circ x_3) + x_3 \circ (\overline{Px_2} \circ x_1) - Px_2 \circ (\overline{x_1} \circ x_3)$$

where $x_i = \begin{pmatrix} \alpha_i & \mathbf{a}_i \\ \mathbf{b}_i & \beta_i \end{pmatrix}$ ($i = 1, 2, 3$).

By straightforward calculations, for the case of $H_3(\mathbf{O})$, we have the following theorem, but long proof, we omit it (see, the details [17], [19]).

Theorem 1.4 *Under the notations as in above, the vector space $(M(V), < xyz >)$ is a balanced Freudenthal-Kantor triple system with $\varepsilon = 1$, $\delta = 1$.*

For balanced Freudenthal-Kantor triple systems, we have the property of Propositions 1.1, 1.2 and 1.3, because balanced is a special case of (ε, δ) -Freudenthal-Kantor triple system, that is, either the case of $\varepsilon = 1$, $\delta = 1$ or $\varepsilon = -1$, $\delta = -1$.

From now on, we assume that Φ is an algebraically closed field of characteristic zero, unless otherwise specified. Thus we can construct the simple exceptional Lie algebras, the Lie triple systems as follows and bisymmetric spaces are described by next section.

$X = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ (balanced case, that is, $\dim M(V) = 56$, $\dim L_{-2} = \dim L_2 = \dim \{K(x, y) | x, y \in M(V)\}_{\text{span}} = 1$), where $\mathbf{a}, \mathbf{b} \in H_3(\mathbf{O})$ (exceptional Jordan algebra with 27 dimension) and $\alpha, \beta \in \Phi$. The Lie algebra constructed from this triple system $U = M(V)$ is the following; $L(U) \cong E_8$, $L_{-2} \oplus L_0 \oplus L_2 \cong E_7 \oplus A_1$, $L_0 \cong E_7 \oplus gl(1)$, $U = M(V) = L_{-1}$.

To change the notation $H_3(\mathbf{O}) \rightarrow H_3(\mathfrak{A})(= B)$, here $\mathfrak{A}$ is Hurwitz algebras over Φ , let $\begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in \begin{pmatrix} \Phi & B \\ B & \Phi \end{pmatrix} = A$, with respect to the $\dim B$, and Lie algebras $L(A)$ obtained from B , we have the following.

$\dim B$	1	6	9	15	27
$\dim A$	4	14	20	32	56
$\dim L(A)$	14	52	78	133	248
$L(A)$	G_2	F_4	E_6	E_7	E_8

where $\dim \mathfrak{A} = 8, 4, 2$ and 1 mean that algebras $\mathfrak{A}$ are the octonion algebra $\mathbf{O}$ over the field Φ , a quaternion $\mathbf{H}$ over Φ , a separable quadratic field over Φ , and the field Φ , respectively. Also, $\dim B = 1$ means that B is the field Φ .

For this case's E_8 , considering the extended Dynkin diagram, we have $L_{-1} \oplus L_1 \cong L(U)/(L_{-2} \oplus L_0 \oplus L_2) = E_8/(A_1 \oplus E_7)$ with $\dim L_{-1} \oplus L_1 = 112$;
 $\odot - \square - \circ - \circ - \circ - \circ - \circ - \circ - \circ$, $\square$ omitted $\cong A_1 \oplus E_7 = \text{Der}(L_{-1} \oplus L_1)$.
 $\quad \quad \quad | \quad \quad \quad \odot$ is the highest root
 $\quad \quad \quad \circ$

Remark This construction of balanced type with $\dim M(V) = 56$ has been first studied by H. Freudenthal (called a metasymmetric geometry equipped with notations of $P \times Q$ and $\{P, Q\}$). That is, this concept is characterized from the triple system (or a ternary algebra) and this $U = M(V)$ is called a generalized Zorn's vector matrix ([13]-[16],[18],[35] and the references of therein).

Remark ([16]) Note that for simple exceptional balanced Freudenthal-Kantor triple systems $U = M(V)$ ($\varepsilon = 1, \delta = 1$), the Lie triple systems $T(U)$ and the standard embedding Lie algebras $L(U)$, if $\dim U = n$, we have the following;

$$\begin{aligned} \dim \text{Der } U &= 3n(n+1)/(n+16), \\ \dim \text{Anti} - \text{Der } U &= \dim \{L(x, y) - L(y, x) | x, y \in U\}_{\text{span}} = 1, \\ \dim T(U) &= 2n, \text{ and } \dim L(U) = (5n^2 + 38n + 48)/(n+16). \end{aligned}$$

Here $L(U)$ are G_2, F_4, E_6, E_7, E_8 types. For example, if $n = 56$, then we obtain $\dim \text{Der } U = 133$ and $\dim L(U) = 248$.

2 Bisymmetric spaces associated with exceptional simple Lie algebras

Following the results due to O. Loos ([51]) or W. Bertram ([5]) with respect to symmetric spaces, it is known that any symmetric space G/H may be identified with some Lie triple system T (as the tangent space of the symmetric space is a Lie triple system).

We consider a concept of bisymmetric space $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ on Lie triple subsystems pair defined as follows:

From the balanced type of exceptional cases, we may define a concept of the pair $(T_\alpha, T_\beta, T_\gamma, T_\delta)$.

$$(\heartsuit) \quad (\dim T_\delta + 16)/\dim T_\gamma = (\dim T_\gamma + 16)/\dim T_\beta = \\ (\dim T_\beta + 16)/\dim T_\alpha = 2, \text{ and } T_\alpha < T_\beta < T_\gamma < T_\delta$$

It is said to be a *quaternionic bisymmetric space*. This means that

$$(\heartsuit\heartsuit) \quad F_4/(C_3 \oplus A_1) < E_6/(A_5 \oplus A_1) < E_7/(D_6 \oplus A_1) < E_8/(E_7 \oplus A_1)$$

and the dimension of bisymmetric spaces of balanced type;

$$28, \quad 40, \quad 64, \quad 112 \quad \text{respectively.}$$

With respect to extended Dynkin diagrams (symmetric spaces) of E_7, E_6, F_4 ;
 $\circ - \circ - \circ - \circ - \circ - \square - \odot$, $\square$ omitted $\cong D_6 \oplus A_1$, and $\odot$ is the highest root.

$$\begin{array}{c} | \\ \circ \\ \circ - \circ - \circ - \circ - \circ - \square - \odot, \quad \square \text{ omitted } \cong A_5 \oplus A_1, \text{ and } \odot \text{ is the highest root.} \\ | \\ \square \\ | \\ \odot \end{array}$$

$$\odot - \square - \circ \Rightarrow \circ - \circ, \quad \square \text{ omitted } \cong C_3 \oplus A_1, \text{ and } \odot \text{ is the highest root.}$$

Here $A_1, A_5, B_4, C_3, D_5, D_6$ mean classical simple Lie algebras.

From now on, we call a quaternionic symmetric space for the balanced case, in the sence of W. Bertram ([6]).

In final results of this section, we deal with the construction of Lie algebra of G_2 type and the bisymmetric space associated with the balanced Freudenthal-Kantor triple system.

Let

$$M(V) = \left\{ \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in V = \Phi \right\},$$

$B(\alpha, \beta) = \alpha\beta$ and the cross product $\times$ be identically zero. Then by Theorem 1.4, we can obtain the structure of the balanced triple system on $M(V)$. If $\langle x, y \rangle$ is nondegenerate, then from the dimensionality of the standard embedding Lie algebra $L(M(\Phi))$ with $L_{-1} = M(V)$, it follows that the Lie algebra is G_2 type.

Indeed, we have $\dim_{\Phi} \langle L(x, y) \rangle_{\text{span}} = 4$, and $K(x, y) = L(y, x) - L(x, y)$, so $\dim_{\Phi} \langle K(x, y) \rangle_{\text{span}} = 1$, since $K(x_1, x_2) = \langle x_1, x_2 \rangle = \beta_1\alpha_2 - \alpha_1\beta_2 + \gamma_1\delta_2 - \delta_1\gamma_2$.

Next, in above $M(V)$, let $\gamma = \delta = 0$, then by Eq. ($\spadesuit$) in section 1, we have

$$\langle x_1 x_2 x_3 \rangle = \begin{pmatrix} 2\alpha_1\beta_2\alpha_3 + \alpha_2\beta_1\alpha_3 & 0 \\ 0 & -(2\alpha_2\beta_1\beta_3 + \alpha_1\beta_2\beta_3) \end{pmatrix},$$

$$\text{for } x_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix}, \quad (i = 1, 2, 3).$$

From $K(x_1, x_2) = \langle x_1, x_2 \rangle = \beta_1\alpha_2 - \alpha_1\beta_2$, we have the fact that the standard embedding Lie algebra $L(M(\Phi))$ is A_2 type.

Hence, we can obtain a bisymmetric space such that $\dim T_{\delta} = 2\dim T_{\gamma}$;

$$T_{\gamma} = A_2 / (A_1 \oplus gl(1)) < T_{\delta} = G_2 / (A_1 \oplus A_1).$$

The extended Dynkin diagram of $Der T_{\delta}$ is the following.

$\odot - \square \equiv \circ$, $\square$ omitted $\cong A_1 \oplus A_1$. and $\odot$ is highest root.

3 Bisymmetric spaces associated with classical simple Lie algebras

In this section, we consider bisymmetric spaces associated with balanced Freu-denthal-Kantor triple systems of classical type.

(#) A_n type (the dimension is $n(n+2)$):

Let $M_A(V)$ be the set

$$M_A(V) := \left\{ \begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{b} & 0 \end{pmatrix} \mid \mathbf{a}, \mathbf{b} \in V = Mat(1, n-1) \right\}, \begin{pmatrix} 0 & \mathbf{a} \\ {}^t\mathbf{b} & 0 \end{pmatrix} \in Mat(n, n; \Phi).$$

We may introduce a triple product as well as exceptional cases as follows:

$$(\spadesuit\spadesuit) \quad < x_1 x_2 x_3 > = x_1 \circ (\overline{P x_2} \circ x_3) + x_3 \circ (\overline{P x_2} \circ x_1) - P x_2 \circ (\overline{x_1} \circ x_3)$$

$$\text{where } x_1 = \begin{pmatrix} 0 & \mathbf{a}_1 \\ \mathbf{a}_2 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & \mathbf{b}_1 \\ \mathbf{b}_2 & 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & \mathbf{c}_1 \\ \mathbf{c}_2 & 0 \end{pmatrix} \in M_A(V).$$

Here the product $\circ$ is defined by $x \circ y = \begin{pmatrix} 0 & \mathbf{a}_1 \\ \mathbf{a}_2 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \mathbf{b}_1 \\ \mathbf{b}_2 & 0 \end{pmatrix} = \begin{pmatrix} B(\mathbf{a}_1, \mathbf{b}_2) & 0 \\ 0 & B(\mathbf{a}_2, \mathbf{b}_1) \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{c}_1 \\ \mathbf{c}_2 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \mathbf{b}_1 \\ \mathbf{b}_2 & 0 \end{pmatrix} = \begin{pmatrix} B(\mathbf{a}_1, \mathbf{b}_2) & 0 \\ 0 & B(\mathbf{a}_2, \mathbf{b}_1) \end{pmatrix} = \begin{pmatrix} 0 & B(\mathbf{a}_2, \mathbf{b}_1)\mathbf{c}_1 \\ B(\mathbf{a}_1, \mathbf{b}_2)\mathbf{c}_2 & 0 \end{pmatrix} \in M_A(V)$, since $B(\mathbf{a}_2, \mathbf{b}_1)\mathbf{c}_1, B(\mathbf{a}_1, \mathbf{b}_2)\mathbf{c}_2 \in Mat(1, n-1; \Phi)$. And we denote $B(\mathbf{a}_1, \mathbf{b}_2) = \mathbf{a}_1 {}^t\mathbf{b}_2 = \mathbf{b}_2 {}^t\mathbf{a}_1 \in Mat(1, 1; \Phi) = \Phi$, also, ${}^t\mathbf{a}$ denotes the transepose matrix of $\mathbf{a}$ and the cross product $\times$ is identically zero, $\overline{x_1} = x_1$, since the diagonal elements of the vector matrix $M_A(V)$ are 0, the notation P is same in section one. By straightfoward calculations, we have

$$(L(x_1, x_2) + L(x_2, x_1))x_3 = \begin{pmatrix} 0 & -\mathbf{c}_1 {}^t\mathbf{d} \\ \mathbf{c}_2 \mathbf{d} & 0 \end{pmatrix}, (L(x_1, x_2) - L(x_2, x_1))x_3 = \begin{pmatrix} 0 & -\alpha \mathbf{c}_1 \\ \alpha \mathbf{c}_2 & 0 \end{pmatrix}, \text{ where } \mathbf{d} = 2 {}^t\mathbf{b}_1 \mathbf{a}_2 + 2 {}^t\mathbf{a}_1 \mathbf{b}_2 + (\mathbf{a}_1 {}^t\mathbf{b}_2 + \mathbf{b}_1 {}^t\mathbf{a}_2) I_{n-1, n-1} \in Mat(n-1, n-1; \Phi) \text{ and } \alpha = \mathbf{a}_1 {}^t\mathbf{b}_2 - \mathbf{b}_1 {}^t\mathbf{a}_2 \in \Phi.$$

Theorem 3.1 *The triple system $M_A(V)(=U)$ defined by $(\spadesuit\spadesuit)$ is a balanced Freudenthal-Kantor triple system with $\varepsilon = 1, \delta = 1$.*

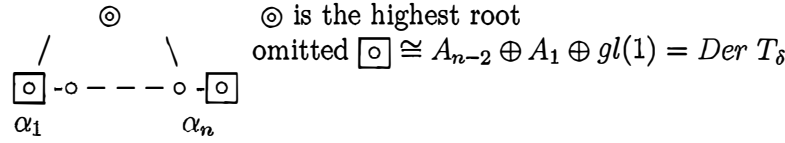
Proof For $\varepsilon = 1$, $\delta = 1$, it is enough to verify the relations (4) and (5) for the case of balanced property ($K(x, y) = \langle x, y \rangle Id$). These relations are straightforward calculations, but we omit it, as it is long. $\square$

Therefor, we get $\langle L(x, y) \rangle_{span} \cong A_{n-2} \oplus gl(1) \oplus gl(1)$ (as a Lie algebra), and $K(x_1, x_2) = -L(x_1, x_2) + L(x_2, x_1) = -\alpha = -(\mathbf{a}_1^t \mathbf{b}_2 - \mathbf{b}_1^t \mathbf{a}_2) \in \Phi$. Hence, the standard embedding Lie algebra implies that $L(U) \cong A_n$ type's Lie algebra associated with $U = M_A(V)$, because $dim_{\Phi} L(U) = dim_{\Phi} \langle L(x, y) \rangle_{span} + 2 dim_{\Phi} \langle K(x, y) \rangle_{span} + 2 dim_{\Phi} M_A(V) = (n-1)^2 + 1 + 2 + 4(n-1) = n(n+2)$. From this A_n , we may construct a bisymmetric space's series ($T_{\alpha}, T_{\beta}, T_{\gamma}, T_{\delta}$), that is, there is a Lie triple subsystem's series as follows:

If $n = 2m - 1$, $m = 2k - 1$, $k = 2l - 1$, then we have a bisymmetric space's series;

$T_{\alpha} = A_l / (A_{l-2} \oplus A_1 \oplus gl(1)) < T_{\beta} = A_k / (A_{k-2} \oplus A_1 \oplus gl(1)) < T_{\gamma} = A_m / (A_{m-2} \oplus A_1 \oplus gl(1)) < T_{\delta} = A_l / (A_{n-2} \oplus A_1 \oplus gl(1)) <$ and $dim T_{\delta} = 2dim T_{\gamma} = 4dim T_{\beta} = 8dim T_{\alpha}$, where $dim T_{\delta} = 4(n-1)$, $dim T_{\gamma} = 4(m-1)$, $dim T_{\beta} = 4(k-1)$, and $dim T_{\alpha} = 4(l-1)$.

Based on the Dynkin diagram of Lie algebra A_n type, the extended Dynkin diagram of corresponding $Der T_{\delta}$ is the following.



($\#\#$) B_n and D_n types (the dimension is $n(2n+1)$ and $n(2n-1)$, respectively):

$$M_{B,D}(V) := \left\{ \begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} \mid \mathbf{a} \in V = Mat(2, p) \right\}, \begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} \in Mat(p+2, p+2; \Phi).$$

We may introduce a triple product as well as exceptional cases as follows:

$$(\spadesuit\spadesuit) \quad \langle x_1 x_2 x_3 \rangle = x_1 \circ (\overline{P x_2} \circ x_3) + x_3 \circ (\overline{P x_2} \circ x_1) - P x_2 \circ (\overline{x_1} \circ x_3)$$

$$\text{where } x_1 = \begin{pmatrix} 0 & \mathbf{a}_1 \\ \mathbf{a}_1 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & \mathbf{b}_1 \\ \mathbf{b}_1 & 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & \mathbf{c}_1 \\ \mathbf{c}_1 & 0 \end{pmatrix} \in M_{B,D}(V).$$

Here the product $\circ$ is defined by $x \circ y = \begin{pmatrix} 0 & \mathbf{a}_1 \\ \mathbf{a}_1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \mathbf{b}_1 \\ \mathbf{b}_1 & 0 \end{pmatrix} = \begin{pmatrix} {}^t(\sigma_0 B(\mathbf{a}_1, \mathbf{b}_1)) & 0 \\ 0 & B(\mathbf{b}_1, \mathbf{a}_1) \sigma_0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{c}_1 \\ \mathbf{c}_1 & 0 \end{pmatrix} \circ \begin{pmatrix} B(\mathbf{a}_1, \mathbf{b}_1) & 0 \\ 0 & B(\mathbf{b}_1, \mathbf{a}_1) \end{pmatrix} = \begin{pmatrix} 0 & B(\mathbf{b}_1, \mathbf{a}_1) \mathbf{c}_1 \\ B(\mathbf{a}_1, \mathbf{b}_1) \mathbf{c}_1 & 0 \end{pmatrix} \in M(V)$, since $B(\mathbf{a}_1, \mathbf{b}_1) \mathbf{c}_1 \in \text{Mat}(2, p; \Phi)$, and $B(\mathbf{b}_1, \mathbf{a}_1) \mathbf{c}_1 \in \text{Mat}(2, p; \Phi)$. We denote the notations by $B(\mathbf{a}_1, \mathbf{b}_1) = \mathbf{a}_1 {}^t \mathbf{b}_1$, $B(\mathbf{b}_1, \mathbf{a}_1) = \mathbf{b}_1 {}^t \mathbf{a}_1 \in \text{Mat}(2, 2; \Phi)$, also, ${}^t \mathbf{a}$ denotes the transepose matrix of $\mathbf{a}$ and the cross product $\times$ is identically zero, $\overline{x_1} = x_1$, $\sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By straightforward calculations, we have the following.

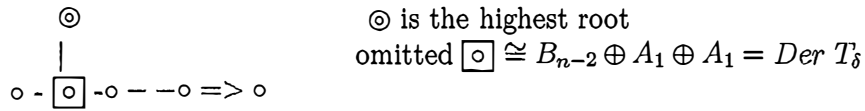
Theorem 3.2 *Then the triple system $M_{B,D}(V)$ defined by $(\spadesuit\spadesuit)$ is a balanced Freudenthal-Kantor triple system with $\varepsilon = 1, \delta = 1$.*

Furthermore, we have $L(x_1, x_2)x_3 = \begin{pmatrix} 0 & \mathbf{c}_1 \mathbf{d} \\ \mathbf{c}_1 \mathbf{d} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{e} \mathbf{c}_1 \\ \mathbf{e} \mathbf{c}_1 & 0 \end{pmatrix}$, where $\mathbf{d} = -({}^t \mathbf{b}_1 \sigma_0 \mathbf{a}_1 + {}^t \mathbf{a}_1 \sigma_0 \mathbf{b}_1) \in \text{Mat}(p, p; \Phi)$, $\mathbf{e} = -\mathbf{a}_1 {}^t \mathbf{b}_1 \sigma_0 \in \text{Mat}(2, 2; \Phi)$. Hence, $\dim_{\Phi} \langle L(x, y) \rangle_{\text{span}} = p(p-1)/2 + 4$, $\dim_{\Phi} \langle K(x, y) \rangle_{\text{span}} = 1$. Therefore, if $p = 2n - 3$, then $\langle L(x, y) \rangle_{\text{span}} \cong B_{n-2} \oplus A_1 \oplus \text{gl}(1)$ (as Lie algebra), $\langle K(x, y) \rangle_{\text{span}} \cong \text{gl}(1) = \Phi$ and the standard embedding Lie algebra $L(M_{B,D}(V))$ is the B_n type. If $p = 2n - 4$, we get $\langle L(x, y) \rangle_{\text{span}} \cong D_{n-2} \oplus A_1 \oplus \text{gl}(1)$ (as a Lie algebra), and $K(x_1, x_2) = -L(x_1, x_2) + L(x_2, x_1) = \langle x, y \rangle \in \Phi$. Hence, the standard embedding Lie algebra $L(M_{B,D}(V))$ is the D_n type. From these B_n and D_n types, we may construct a bisymmetric space's series $(T_{\alpha}, T_{\beta}, T_{\gamma}, T_{\delta})$, that is, Lie triple subsystems as follows:

If $n = 3(m - 1)$, $m = 3(k - 1)$, $k = 3(l - 1)$ (the case of B_n), then we have bisymmetric spaces satisfying

$T_{\alpha} = B_l / (B_{l-2} \oplus A_1 \oplus A_1) < T_{\beta} = B_k / (B_{k-2} \oplus A_1 \oplus A_1) < T_{\gamma} = B_m / (B_{m-2} \oplus A_1 \oplus A_1) < T_{\delta} = B_n / (B_{n-2} \oplus A_1 \oplus A_1)$ and $\dim T_{\delta} = 3 \dim T_{\gamma} = 9 \dim T_{\beta} = 27 \dim T_{\alpha}$, where $\dim T_{\delta} = 4(2n - 3)$, $\dim T_{\gamma} = 4(2m - 3)$, $\dim T_{\beta} = 4(2k - 3)$, and $\dim T_{\alpha} = 4(2l - 3)$.

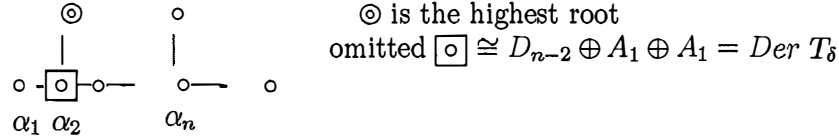
The extended Dynkin diagram of corresponding $\text{Der } T_{\delta}$ is the following.



α_1 α_n

If $n = 2(m-1)$, $m = 2(k-1)$, $k = 2(l-1)$ (the case of D_n), then we have a quaternionic bisymmetric spaces satisfying

$T_\alpha = D_l/(D_{l-2} \oplus A_1 \oplus A_1) < T_\beta = D_k/(D_{k-2} \oplus A_1 \oplus A_1) < T_\gamma = D_m/(D_{m-2} \oplus A_1 \oplus A_1) < T_\delta = D_n/(D_{n-2} \oplus A_1 \oplus A_1) <$ and $\dim T_\delta = 2\dim T_\gamma = 4\dim T_\beta = 8\dim T_\alpha$. The extended Dynkin diagram of corresponding $\text{Der } T_\delta$ is the following.



(###) C_n type (the dimension is $n(2n+1)$):

$M_C(V) := \left\{ \begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} \mid \mathbf{a} \in V = \text{Mat}(1, 2(n-1)) \right\}$, $\begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} \in \text{Mat}(2n-1, 2n-1; \Phi)$. We may introduce a triple product as well as exceptional cases as follows:

$$(\spadesuit\spadesuit) \quad \langle x_1 x_2 x_3 \rangle = x_1 \circ (\overline{P}x_2 \circ x_3) + x_3 \circ (\overline{P}x_2 \circ x_1) - Px_2 \circ (\overline{x}_1 \circ x_3)$$

where $x_1 = \begin{pmatrix} 0 & \mathbf{a}_1 \\ \mathbf{a}_1 & 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 0 & \mathbf{b}_1 \\ \mathbf{b}_1 & 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 0 & \mathbf{c}_1 \\ \mathbf{c}_1 & 0 \end{pmatrix} \in M_C(V)$.

Here the product $\circ$ is defined by $x \circ y = \begin{pmatrix} 0 & \mathbf{a}_1 \\ \mathbf{a}_1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \mathbf{b}_1 \\ \mathbf{b}_1 & 0 \end{pmatrix} = \begin{pmatrix} B(\mathbf{a}_1, \mathbf{b}_1) & 0 \\ 0 & B(\mathbf{b}_1, \mathbf{a}_1) \end{pmatrix}$, $\begin{pmatrix} 0 & \mathbf{c}_1 \\ \mathbf{c}_1 & 0 \end{pmatrix} \circ \begin{pmatrix} B(\mathbf{a}_1, \mathbf{b}_1) & 0 \\ 0 & B(\mathbf{b}_1, \mathbf{a}_1) \end{pmatrix} = \begin{pmatrix} 0 & B(\mathbf{b}_1, \mathbf{a}_1)\mathbf{c}_1 \\ B(\mathbf{a}_1, \mathbf{b}_1)\mathbf{c}_1 & 0 \end{pmatrix} \in M(V)$, since $B(\mathbf{a}_1, \mathbf{b}_1)\mathbf{c}_1$ and $B(\mathbf{b}_1, \mathbf{a}_1)\mathbf{c}_1 \in \text{Mat}(1, 2(n-1); \Phi)$. And we denote $B(\mathbf{a}_1, \mathbf{b}_1) = \frac{1}{2} \langle \mathbf{a}_1, \mathbf{b}_1 \rangle$, $B(\mathbf{b}_1, \mathbf{a}_1) = -B(\mathbf{a}_1, \mathbf{b}_1) \in \text{Mat}(1, 1; \Phi) = \Phi$, also, the cross product $\times$ is identically zero, $\overline{x}_1 = x_1$. By straightforward calculations, we have the following.

Theorem 3.3 *The triple system $M_C(V)$ defined by $(\spadesuit\spadesuit)$ is a balanced Freu-*

dential-Kantor triple system with $\varepsilon = 1$, $\delta = 1$.

Furthermore, we have $L(x_1, x_2)x_3 = \begin{pmatrix} 0 & \mathbf{d} \\ \mathbf{d} & 0 \end{pmatrix}$, where $\mathbf{d} = -B(\mathbf{c}_1, \mathbf{b}_1)\mathbf{a}_1 - B(\mathbf{a}_1, \mathbf{b}_1)\mathbf{c}_1 - B(\mathbf{c}_1, \mathbf{a}_1)\mathbf{b}_1 \in \text{Mat}(1, 1; \Phi) = \Phi$. That is, we get

$\langle L(x, y) \rangle_{\text{span}} \cong C_{n-1} \oplus gl(1)$ (as Lie algebra), and $K(x, y) = -L(x, y) + L(y, x) = \langle x, y \rangle Id = gl(1) = \Phi$. Hence, $\dim_{\Phi} \langle L(x, y) \rangle_{\text{span}} = \dim_{\Phi} \langle K(x, y) \rangle_{\text{span}} = 1$, therefor, the standard embedding Lie algebra $L(M_C(V))$ is the C_n type.

From this C_n , we may construct a bisymmetric space's series $(T_\alpha, T_\beta, T_\gamma, T_\delta)$, that is, Lie triple subsystems as follows:

If $n = 2(m-1)$, $m = 2(k-1)$, $k = 2(l-1)$ (the case of B_n), then we have a quaternionic bisymmetric spaces satisfying

$T_\alpha = C_l / (C_{l-1} \oplus A_1) < T_\beta = C_k / (C_{k-1} \oplus A_1) < T_\gamma = C_m / (C_{m-1} \oplus A_1) < T_\delta = C_n / (C_{n-1} \oplus A_1)$ and $\dim T_\delta = 2\dim T_\gamma = 4\dim T_\beta = 8\dim T_\alpha$, where $\dim T_\delta = 4(n-1)$, $\dim T_\gamma = 4(m-1)$, $\dim T_\beta = 4(k-1)$, and $\dim T_\alpha = 4(l-1)$.

The extended Dynkin diagram of corresponding $Der T_\delta$ is the following.
 $\odot \Rightarrow \boxed{\odot} - \circ - \circ - \circ \Leftarrow \circ$ omitted $\boxed{\odot} \cong C_{n-1} \oplus A_1 = Der T_\delta$, $\odot$ is highest root.

α_1 α_n

For the Lie triple systems pair $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ created by these methods, we note to call a "quaternionic bisymmetric space" associated with the balanced triple systems constructed in section 2 and 3.

Remark The concept of bisymmetric spaces can be continued without stopping for classical types if $n \gg 0$.

In final comments of this section, with respect to the construction of quaternionic symmetric spaces and simple Lie algebras equipped with the $sl(2)$ -module obtained from $\langle f, g, h \rangle_{\text{span}} \cong A_1$, we emphasize that the construction created from balanced Freudenthal-Kantor triple systems is an important concept without using root systems. On the other hand, in ([42], [43], [44] and the references of therein), our subjects are considered by other triple systems equipped with no balanced types or by normal triality algebras (that is, by means of nonassociative algebras, in particular, by generalized vector matrix without using balanced cases).

In next sections, we will study about certain bisymmetric superspaces obtained from Lie superalgebras with associated with $(\varepsilon, -1)$ Freudenthal-Kantor triple systems.

4 Bisymmetric superspaces associated with balanced $(-1, -1)$ -Freudenthal-Kantor triple systems

In this section, we consider bisymmetric superspaces associated with balanced Freudenthal-Kantor triple systems equipped with $\varepsilon = -1$ and $\delta = -1$ as well as section 2 and section 3 (the case of $\varepsilon = 1, \delta = 1$). The case of $T(-1, -1)$ associated with a $(-1, -1)$ -Freudenthal-Kantor triple system is an anti-Lie triple system and so this means a symmetric superspace. For example, following concept of section 1 and section 2, it is known that there is a correspondence the symmetric space and the Lie triple system, because the tangent space of a symmetric space is a Lie triple system. Hence, we can generalize this concept to a correspondence of the symmetric superspace and the anti-Lie triple system.

Hereby for Lie superalgebras, we use notations as follows;

$A(m-1, n-1) = sl(m|n)$, $B(m, n) = osp(2m+1|2n)$, $C(n+1) = osp(2|2n)$, $D(m, n) = osp(2m|2n)$, in particular, $B(0, n) = osp(1|2n)$.

(####) $B(n, 1)$ types (the dimension is $(2n^2 + 5n + 5)$) and $D(n, 1)$ (the dimension is $(2n^2 + 3n + 1)$) :

Let V be a vector space of the matrix set of $Mat(1, l; \Phi)$ and $\langle x, y \rangle = x^t y \in \Phi^*$, $\forall x, y \in V$. Then we may define a triple product as follows;

$$(\spadesuit\spadesuit\spadesuit) \quad \langle xyz \rangle = \frac{1}{2}(\langle x, y \rangle z - \langle y, z \rangle x + \langle z, x \rangle y), \quad \forall x, y, z \in V.$$

Theorem 4.1 *The triple system V defined by $(\spadesuit\spadesuit\spadesuit)$ is a balanced Freudenthal-Kantor triple system with $\varepsilon = -1$ and $\delta = -1$.*

Indeed, by straightforward calculations, we have the relations $L(x, y) + L(y, x) = K(x, y) = \langle x, y \rangle Id$ and Eq.(4), that is, a $(-1, -1)$ -Freudenthal-Kantor triple system equipped with $\dim_{\Phi} \langle K(x, y) \rangle_{span} = 1$.

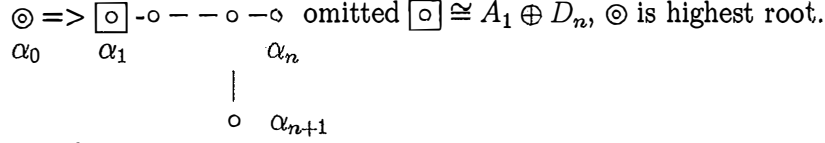
• $l = 2n+1 \rightarrow$ the standard embedding Lie superalgebra $L(V)$ is $B(n, 1)$ type.

• $l = 2n \rightarrow$ the standard embedding Lie superalgebra $L(V)$ is $D(n, 1)$ type.

For these extended Dynkin diagrams of $L(V)$, we have

$$\odot \Rightarrow \boxed{\circ} - \circ - \circ \Rightarrow \circ \text{ omitted } \boxed{\circ} \cong A_1 \oplus B_n, \odot \text{ is highest root.}$$

$$\alpha_0 \quad \alpha_1 \quad \alpha_{n+1}$$



Thus we get two bisymmetric superspaces as follows;

$$T_\delta = B(n, 1)/(A_1 \oplus \mathbf{B}_n) \quad (\text{the dimension is } 2(2n + 1)),$$

$$T_\delta = D(n, 1)/(A_1 \oplus D_n) \quad (\text{the dimension is } 2(2n)).$$

Therefore, for $B(n, 1)$ if $n = 3m + 1$, $m = 3l + 1$, $l = 3k + 1$, then we have

$$T_\delta > T_\gamma > T_\beta > T_\alpha \text{ and } \dim T_\delta = 3\dim T_\gamma = 9\dim T_\beta = 27\dim T_\alpha.$$

Also, for the case of $D(n, 1)$, by $T_\delta = D(n, 1)/(A_1 \oplus D_n)$, if $n = 2m$, $m = 2l$, $l = 2k$, then similarly we have

$$T_\delta > T_\gamma > T_\beta > T_\alpha \text{ and } \dim T_\delta = 2\dim T_\gamma = 4\dim T_\beta = 8\dim T_\alpha.$$

(#####) $A(n - 1, 1) = sl(n, 2)$ type (the dimension is $(n^2 + 4n + 3)$):

Let V be a vector space of the matrix set of $Mat(1, 2n; \Phi)$ and with an anti-symmetric bilinear form $\langle x, y \rangle \in \Phi^*$, $\forall x, y \in V$, similarly to section 3, denote by $A := M(V) = \left\{ \begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} \mid \mathbf{a} \in V = Mat(1, 2n; \Phi) \right\}$.

Then we may define a triple product as follows;

$$(\spadesuit\spadesuit\spadesuit\spadesuit) \langle x_1 x_2 x_3 \rangle =$$

$$x_1 \circ (Px_2 \circ x_3) - x_3 \circ (Px_2 \circ x_1) + Px_2 \circ (x_1 \circ x_3), \quad \forall x_i \ (i = 1, 2, 3) \in A.$$

Theorem 4.2 *The triple system A defined by $(\spadesuit\spadesuit\spadesuit\spadesuit)$ is a balanced Freudenthal-Kantor triple system with $\varepsilon = -1$ and $\delta = -1$.*

Indeed, by straightforward calculations, we have the relatins $L(x, y) + L(y, x) = K(x, y) = \langle x, y \rangle Id$ and Eq.(4), these imply that A is a $(-1, -1)$ -Freudenthal-Kantor triple system equipped with $\dim_\Phi \{K(x, y)\}_{span} = 1$. Hence, from this result of $L_{-1} = A$ (balanced triple system), the standard embedding Lie superalgebra $L(A)$ is obtained $A(n - 1, 1) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ type, so we have $L_{-1} \oplus L_1 = T(A)$ (anti-Lie triple system), and $L_{-2} \oplus L_0 \oplus L_2 = Der T(A)$ (the derivation of $T(A)$). The dimensions are $\dim L(A) = n^2 + 4n + 3$, $\dim L_{-1} = \dim L_1 = 2n$, $\dim L_{-2} = \dim L_2 = 1$, and $\dim L_0 = \dim A_{n-1} + \dim (gl(1) \oplus gl(1)) = n^2 + 1$.

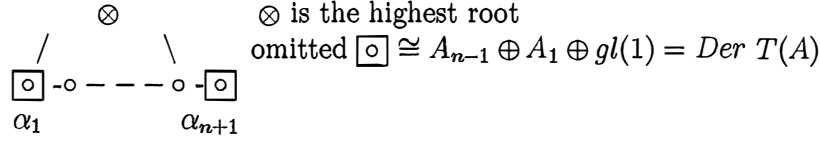
Thus, we can obtain bisymmetric superspaces associated with the balanced triple system as follows;

If $n = 2m$, $m = 2l$, $l = 2k$,

$$T_\delta = A(n-1, 1)/(A_1 \oplus A_{n-1} \oplus gl(1)), \dots, T_\alpha = A(k-1, 1)/(A_1 \oplus A_{k-1} \oplus gl(1))$$

and $T_\delta > T_\gamma > T_\beta > T_\alpha$, furthermore, $\dim T_\delta = 2\dim T_\gamma = 4\dim T_\beta = 8\dim T_\alpha$.

The extended Dynkin diagram of corresponding $Der T(A)$ is the following.



We call a *quadratic bisymmetric superspace's series* $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ with respect to the bisymmetric superspace $T(A)$ associated with the balanced $(-1, -1)$ -Freudenthal-Kantor triple system defined by $(\spadesuit\spadesuit\spadesuit)$ of Theorem 4.1 and $(\spadesuit\spadesuit\spadesuit\spadesuit)$ of Theorem 4.2.

Note that we do not know a construction of the $C(n+1)$ type with the dimension of $n^2 + 5n + 1$ from balanced $(-1, -1)$ -Freudenthal-Kantor triple system, however we can obtain a construction a $(1, -1)$ -Jordan triple system ([42], [44]).

In final of this section, with respect to the bisymmetric superspaces induced from Lie superalgebras $F(4)$ and $G(3)$ types, to describe it briefly, we need the following results. According to ([7], [26]), for the construction of simple Lie superalgebras $F(4)$ with 40 dimension and $G(3)$ with 31 dimension, based upon balanced $(-1, -1)$ -FKTS U equipped with $\dim U = 8$ (the case of $F(4)$) and $\dim U = 7$ (the case of $G(3)$), furthermore, by means of concept of the anti-Lie triple system $T(U) = U \oplus U$ associated with $U (= L_{-1})$, and of the derivation $Der T(U) = L_{-2} \oplus L_0 \oplus L_2$, we have studied for the construction of 5-graded Lie superalgebra $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ of following two types;

$F(4)$; $\otimes \Rightarrow \boxed{\otimes} - \circ \leq \circ - \circ$, $\boxed{\otimes}$ omitted $\cong A_1 \oplus B_3 = Der T(U)$, $\otimes$ is the highest root.

$G(3)$; $\otimes \equiv > \boxed{\otimes} - \circ \leq \circ$, $\boxed{\otimes}$ omitted $\cong A_1 \oplus G_2 = Der T(U)$, $\otimes$ is the highest root.

Here, the figures are obtained from the extended Dynkin diagram of $F(4)$ and $G(3)$ types.

That is, the former type is induced from the octonion algebra $\mathbf{O}$ over the field Φ and the latter type is induced from $Im \mathbf{O} = \{x \in \mathbf{O} \mid trace x = 0\}$ over Φ , because with respect to the triple product in $U = \mathbf{O}$ or $Im \mathbf{O}$, there is the structure of a balanced $(-1, -1)$ -FKTS in $U = L_{-1}$ and $T(U) = L_{-1} \oplus L_1$ is an anti-LTS, hence we can obtain the standard embedding Lie superalgebras $L(U) = T(U) \oplus Der T(U)$ equipped with $dim L_{-2} = dim L_2 = 1$ satisfying $L(U) = F(4)$ (if $dim U = 8$) and $L(U) = G(3)$ (if $dim U = 7$).

From these results, one can create bisymmetric superspaces $(T_\beta, T_\gamma, T_\delta)$ as follows;

- $osp(2|2)/(A_1 \oplus gl(1)) < osp(1|4)/(A_1 \oplus A_1) < F(4)/(A_1 \oplus B_3)$,
 - $osp(1|2)/A_1 < osp(3|2)/(A_1 \oplus A_1) < G(3)/(A_1 \oplus G_2)$,
- where these mean that $osp(1|4) = B(0, 2)$, $osp(2|2) = C(2)$, $osp(3|2) = B(1, 1)$ and $osp(1|2) = B(0, 1)$, also, these dimensions are $dim B(0, 2) = 14$, $dim C(2) = 8$, $dim B(1, 1) = 12$, and $dim B(0, 1) = 5$, respectively.

Hereby, the construction of $osp(1|4)$ (resp. $osp(3|2)$) is induced from a balanced $(-1, -1)$ FKTS in the quaternion algebra $\mathbf{H}$ (resp. $Im \mathbf{H}$) over Φ , and the construction of $osp(2|2)$ (resp. $osp(1|2)$) is induced from a balanced $(-1, -1)$ FKTS in the separable quadratic field $\mathbf{3}$ (resp. $Im \mathbf{3}$) over Φ . We remark for these cases that if $dim U = 8$, then $4dim T_\beta = 2dim T_\gamma = dim T_\delta$, and if $dim U = 7$, then $2(dim T_\beta + 1) = dim T_\gamma$, and $2(dim T_\gamma + 1) = dim T_\gamma$.

Moreprecisely describing for these cases, we have extended Dynkin diagrams as follows;

- $B(0, 2); \odot \Rightarrow \boxed{\odot} \Rightarrow \bullet$, $\boxed{\odot}$ omitted $\cong A_1 \oplus A_1$, $\odot$ is the highest root,
 $C(2); \otimes \Leftarrow \boxed{\otimes}$, $\boxed{\otimes}$ omitted $\cong A_1 \oplus gl(1)$, $\otimes$ is the highest root,
 $B(1, 1); \odot \Rightarrow \boxed{\otimes} \Rightarrow \circ$, $\boxed{\otimes}$ omitted $\cong A_1 \oplus A_1$, $\odot$ is the highest root, and
 $B(0, 1); \odot \Rightarrow \boxed{\otimes}$, $\boxed{\otimes}$ omitted $\cong A_1$, $\odot$ is the highest root.

5 Other bisymmetric (super)spaces associated with (ε, δ) -FKTS $U(\varepsilon, \delta)$

In this section, we deal with several bisymmetric (super) spaces without possessing the balanced property. That is, there is a lot of the construction

of Lie (super)algebras associated with triple systems without using balanced properties, for example, we have typical four types as follows:

- 1) (ε, δ) - Jordan triple systems, in particular, $\varepsilon = -1$, and $\delta = \pm 1$,
- 2) $\langle xyz \rangle = \langle y, z \rangle x$, where $\langle x, y \rangle = -\varepsilon \langle y, x \rangle$,
- 3) $\langle xyz \rangle = x^t y z + \delta(z^t y x - z^t x y)$, $x, y, z \in \text{Mat}(n, n; \Phi)$, where $^t x$ is the transpose matrix of x ,
- 4) $(-1, 1)$ -Freudenthal-Kantor triple system induced from $\mathbf{O} \otimes \mathbf{O}$.

1)(a); Let U be a set of $\text{Mat}(1, p; \Phi)$. Then $(U, \{xyz\})$ is a Jordan triple system with respect to the product $\{xyz\} = \langle x, y \rangle z + \langle y, z \rangle x - \langle z, x \rangle y$, where $\langle x, y \rangle = \langle y, x \rangle \in \Phi^*$ and so the standard embedding Lie algebra $L(U) = L_{-1} \oplus L_0 \oplus L_0$ associated with $U = L_{-1}$ is a B_{n+1} type (if $p = 2n + 1$) and a D_{n+1} (if $p = 2n$), because $K(x, y) = 0$, $\forall x, y \in U$.

The extended Dynkin diagram is the following;

$$\begin{array}{c} \boxed{\circ} - \circ - \circ - \circ - \circ = \circ, \text{ omitted } \boxed{\circ} \cong B_n \oplus gl(1) = \text{Der } T(U). \\ \alpha_1 \quad | \quad \alpha_{n+1} \\ \odot \alpha_0 \end{array}$$

$$\begin{array}{c} \boxed{\circ} - \circ - \circ - \circ - \circ - \circ, \text{ omitted } \boxed{\circ} \cong D_n \oplus gl(1) = \text{Der } T(U). \\ \alpha_1 \quad | \quad | \quad \alpha_n \\ \odot \alpha_0 \quad \circ \alpha_{n+1} \end{array}$$

Thus we can obtain for the standard embedding Lie algebra of two types. That is, $L(U) = B_{n+1}$ if $p = 2n + 1$ and $L(U) = D_{n+1}$ if $p = 2n$, and $L_{-2} = L_2 = 0$, because $K(x, y) = 0$ (identically).

Hence the bisymmetric spaces of these cases are given by $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying

$$\cdot T_\alpha = B_{k+1}/(B_k \oplus gl(1)) < T_\beta = B_{l+1}/(B_l \oplus gl(1)) < T_\gamma = B_{m+1}/(B_m \oplus gl(1)) < T_\delta = B_{n+1}/(B_n \oplus gl(1)), \text{ and } 27 \dim T_\alpha = 9 \dim T_\beta = 3 \dim T_\gamma = \dim T_\delta, \text{ where } n = 3m + 1, m = 3l + 1, \text{ and } l = 3k + 1.$$

Similarly, for D_{n+1} type, we have

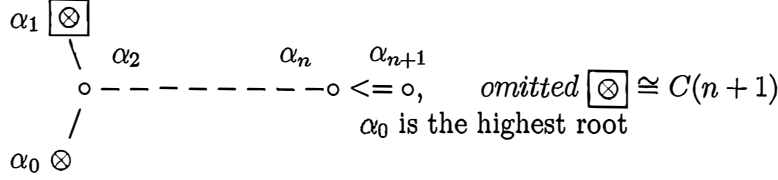
$(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying

$$\cdot T_\alpha = D_{k+1}/(D_k \oplus gl(1)) < T_\beta = D_{l+1}/(D_l \oplus gl(1)) < T_\gamma = D_{m+1}/(D_m \oplus gl(1)) < T_\delta = D_{n+1}/(D_n \oplus gl(1)), \text{ and } 8 \dim T_\alpha = 4 \dim T_\beta = 2 \dim T_\gamma = \dim T_\delta, \text{ where } n = 2m, m = 2l, \text{ and } l = 2k.$$

1)(b); Let U be a set of $\text{Mat}(1, 2n; \Phi)$. Then $(U, \{xyz\})$ is an anti-Jordan triple system ($\varepsilon = 1$ and $\delta = -1$) with respect to the product $\{xyz\} = \langle x, y \rangle z + \langle y, z \rangle x - \langle z, x \rangle y$, where $\langle x, y \rangle = -\langle y, x \rangle \in \Phi^*$ and so

the standard embedding Lie algebra $L(U) = L_{-1} \oplus L_0 \oplus L_1$ associated with $U = L_{-1}$ is a $C(n+1)$ type with $\dim L(U) = 2n^2 + 5n + 1$.

The extended Dynkin diagram is the following;



Note that $L(U) = L_{-1} \oplus L_0 \oplus L_1$, $\text{Der } T(U) = L_0 \cong C_n \oplus gl(1)$, and $L_{-2} = L_2 = 0$, because $K(x, y) = 0$ identically.

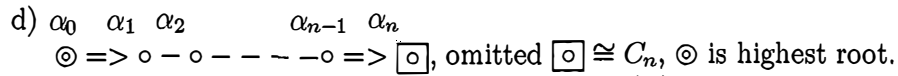
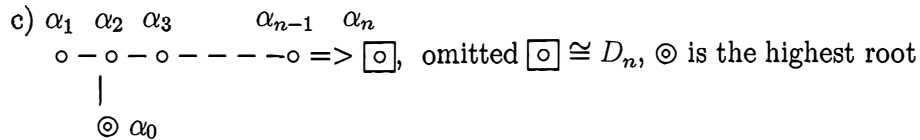
Thus the bisymmetric superspaces of this case are given by $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying

$\cdot T_\alpha = C_{k+1}/(C_k \oplus gl(1)) < T_\beta = C_{l+1}/(C_l \oplus gl(1)) < T_\gamma = C_{m+1}/(C_m \oplus gl(1)) < T_\delta = C_{n+1}/(C_n \oplus gl(1))$, and $8 \dim T_\alpha = 4 \dim T_\beta = 2 \dim T_\gamma = \dim T_\delta$, where $n = 2m, m = 2l$, and $l = 2k$.

2) Let U be a set of $Mat(1, n; \Phi)$. Then $(U, \{xyz\})$ is a $(-1, \delta)$ -FKTS with respect to the product $\{xyz\} = \langle y, z \rangle x$, where $\langle x, y \rangle = \langle y, x \rangle \in \Phi^*$ and so the standard embedding Lie algebra $(\delta = 1)$ is 5-graded B_n type (this is called (2)(c) type), and the standard embedding Lie superalgebra $(\delta = -1)$ is 5-graded $B(0, n)$ type (this is called (2)(d) type), That is, $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ associated with $U = L_{-1}$ is two cases as follows;

$\dim L(U) = n(2n+1)$ if $\delta = 1$, and $K(x, z)y = \langle z, y \rangle x - \langle x, y \rangle z$,
 $\dim L(U) = 2n^2 + 3n$ if $\delta = -1$, and $K(x, z)y = \langle y, x \rangle z + \langle y, z \rangle x$

The extended Dynkin diagrams are the following;



Note that $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, $\text{Der } T(U) = L_{-2} \oplus L_0 \oplus L_2 \cong D_n$ (if $\delta = 1$ of the case (c)) and $\text{Der } T(U) = L_{-2} \oplus L_0 \oplus L_2 \cong C_n$ (if $\delta = -1$ of the case (d)).

The bisymmetric (super)spaces of these cases (c) and (d) are given by

$(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying

c) $T_\alpha = B_k/D_k < T_\beta = B_l/D_l < T_\gamma = B_m/D_m < T_\delta = B_n/D_n$, and $8 \dim T_\alpha = 4 \dim T_\beta = 2 \dim T_\gamma = \dim T_\delta$, where $n = 2m, m = 2k$, and $l = 2k$.

Similarly, for $B(0, n)$ we have

d) $T_\alpha = B(0, k)/C_k < T_\beta = B(0, l)/C_l < T_\gamma = B(0, m)/C_m < T_\delta = B(0, n)/C_n$, and $8 \dim T_\alpha = 4 \dim T_\beta = 2 \dim T_\gamma = \dim T_\delta$, where $n = 2m, m = 2l$, and $k = 2l$.

For the case of which contains $\varepsilon = 1$, we can generalize to the following.

Remark For the bilinear form $\langle x, y \rangle = -\varepsilon \langle y, x \rangle \in \Phi^*$, and introduce the product $\{xyz\} = \langle y, z \rangle x$, then this triple system is a (ε, δ) -Freudenthal-Kantor triple system without balanced property.

3) (e); For the case of $\delta = 1$, let U be a set of $Mat(2n+1, 2n+1; \Phi)$. Then $(U, \{xyz\})$ is a structurable algebra (or a $(-1, 1)$ -FKTS, [1], [31], [32], [33] and references of therein) with respect to the product $\{xyz\} = x^t y z + z^t y x - z^t x y$, $\forall \in U$. Then the Lie algebra $L(U)$ is a 5-graded B_{3n+1} type with $18n^2 + 15n + 3$ dimension. The extended Dynkin diagram is the following;

$\circ - \circ - \circ - \circ - \circ - \boxed{\circ} - \circ - \circ - \circ - \circ = > \circ$, $\boxed{\circ}$ omitted $\cong D_{2n+1} \oplus B_n = Der T(U)$,

$\alpha_1 \mid \alpha_{2n+1} \quad \alpha_{3n+1}$
 $\odot \alpha_0 \quad \alpha_0 (\odot)$ is the highest root.

Next, we consider the case 3) (d)'s case, let U be a set of $Mat(2n, 2n; \Phi)$, then the Lie algebra $L(U)$ is a 5-graded D_{3n} type with $18n^2 - 3n$ dimension. The extended Dynkin diagram is the following;

$\circ - \circ - \circ - \circ - \circ - \boxed{\circ} - \circ - \circ - \circ - \circ$, $\boxed{\circ}$ omitted $\cong D_{2n} \oplus D_n = Der T(U)$,

$\alpha_1 \mid \alpha_{2n} \quad \alpha_{3n-1}$
 $\odot \alpha_0 \quad \alpha_{3n} \quad \alpha_0 (\odot)$ is the highest root.

Hence the bisymmetric spaces of the case (e) is given by $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying if $\dim U = (2n+1)^2$, $Der T(U) = D_{2n+1} \oplus B_n$,

$T_\alpha = B_{k+1}/(D_{2k+1} \oplus B_k) < T_\beta = B_{3l+1}/(D_{2l+1} \oplus B_l) < T_\gamma = B_{3m+1}/(D_{2m+1} \oplus B_m) < T_\delta = B_{3n+1}/(D_{2n+1} \oplus B_n)$, and $729 \dim T_\alpha = 81 \dim T_\beta = 9 \dim T_\gamma = \dim T_\delta$, where $n = 3m+1, m = 3l+1$, and $l = 3k+1$.

Similarly, for D_{3n} type if $\dim U = (2n)^2$, $Der T(U) = D_{2n} \oplus D_n$, we have $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying

• $T_\alpha = D_{3k}/(D_{2k} \oplus D_k) < T_\beta = D_{3l}/(D_{2l} \oplus D_l) < T_\gamma = D_{3m}/(D_{2m} \oplus D_m) < T_\delta = D_{3n}/(D_{2n} \oplus D_n)$, and $64 \dim T_\alpha = 16 \dim T_\beta = 4 \dim T_\gamma = \dim T_\delta$, where $n = 2m, m = 2l$, and $l = 2k$. Note that for 5-graded cases, $T(U) = L_{-1} \oplus L_1$, $Der T(U) = L_{-2} \oplus L_0 \oplus L_2$ and $T(U) = L(U)/Der T(U)$.

3) (f); For the case of $\delta = -1$, let U be a set of $Mat(2n+1, 2n+1; \Phi)$. Then $(U, \{xyz\})$ is a $(-1, -1)$ -FKTS (or an anti-structurable algebra ([31]) with respect to the product

$$\{xyz\} = x^t y z - z^t y x + z^t x y, \quad \forall x, y, z \in U.$$

Then the Lie superalgebra constructed from $U = L_{-1}$ is 5-graded $B(n, 2n+1)$ type with $(18n^2 + 19n + 5)$ dimension.

The extended Dynkin diagram is the following;

$$\odot \Rightarrow \circ - \circ - \cdots - \boxed{\circ} - \circ - \cdots - \circ \Rightarrow \circ, \quad \boxed{\circ} \text{ omitted} \cong C_{2n+1} \oplus B_n,$$

$$\alpha_0 \quad \alpha_1 \quad \alpha_{2n+1} \quad \alpha_{3n+1}$$

Hence we remark that $L_0 = \{L(x, y) | x, y \in U\}_{span} \cong A_{2n} \oplus B_n \oplus gl(1)$.

Next let U be a set of $Mat(2n, 2n; \Phi)$. Then similarly we have the fact that $(U, \{xyz\})$ is a $(-1, -1)$ -FKTS (or an anti-structurable algebra ([31]) with respect to the product

$$\{xyz\} = x^t y z - z^t y x + z^t x y, \quad \forall x, y, z \in U.$$

Then the Lie superalgebra constructed from $U = L_{-1}$ is 5-graded $D(n, 2n)$ type with $(18n^2 + n)$ dimension.

The extended Dynkin diagram is the following;

$\odot \Rightarrow \circ - \circ - \circ - \boxed{\circ} - \circ - \circ - \circ, \boxed{\circ} \text{ omitted} \cong C_{2n} \oplus D_n = \text{Der } T(U),$

$$\begin{array}{cccc|c} \alpha_0 & \alpha_1 & \alpha_{2n} & & \alpha_{3n-1} \\ & & & & \alpha_{3n} \end{array}$$

Hence we remark that $L_0 \cong A_{n-1} \oplus D_n \oplus gl(1)$.

Hence the bisymmetric spaces of the case (f) is given by $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying if $\dim U = (2n+1)^2$, $\text{Der } T(U) = C_{2n+1} \oplus B_n$,

$\cdot T_\alpha = B(k, 2k+1)/(C_{2k+1} \oplus B_k) < T_\beta = B(l, 2l+1)/(C_{2l+1} \oplus B_l) <$
 $T_\gamma = B(m, 2m+1)/(C_{2m+1} \oplus B_m) < T_\delta = B(n, 2n+1)/(C_{2n+1} \oplus B_n)$, and
 $729 \dim T_\alpha = 81 \dim T_\beta = 9 \dim T_\gamma = \dim T_\delta$, where $n = 3m+1, m = 3l+1$,
 and $l = 3k+1$.

Similarly, for $D(n, 2n)$ type if $\dim U = (2n)^2$, $\text{Der } T(U) = C_{2n} \oplus D_n$, we have $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying

$\cdot T_\alpha = D(k, 2k)/(C_{2k} \oplus D_k) < T_\beta = D(l, 2l)/(C_{2l} \oplus D_l) < T_\gamma = D(m, 2m)/(C_{2m} \oplus D_m) < T_\delta = D(n, 2n)/(C_{2n} \oplus D_n)$, and $64 \dim T_\alpha = 16 \dim T_\beta = 4 \dim T_\gamma = \dim T_\delta$, where $n = 2m, m = 2l$, and $l = 2k$. Note that for these 5-graded cases, $T(U) = L_{-1} \oplus L_1$, $Der T(U) = L_{-2} \oplus L_0 \oplus L_2$ and $T(U) = L(U)/Der T(U)$.

Example 5.1 (a generalization of δ structurable algebras) For the case of $U = Mat(m, n; \Phi)$ (if $m \neq n$), note that the triple product

$$\{xyz\} = x^t yz + \delta(z^t yx + z^t xy), \quad \forall x, y, z \in U.$$

is a $(-1, \delta)$ -FKTS, of cause, there is the both cases of $\delta = \pm 1$.

4) (g); Let U be a vector space's set of $\mathbf{O} \otimes \mathbf{O}$ (the dimension of U is 64). Then $(U, < xyz >)$ is a $(-1, 1)$ -FKTS with respect to the product

$$< xyz > = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y, \quad \forall x, y, z \in U,$$

where $\bar{x}$ is the involution satisfying $\bar{\bar{x}} = x$, and so the standard embedding Lie algebra $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is E_8 type with $U = L_{-1}$, $\dim L_{-2} = \dim L_2 = 14$ and $\dim L_0 = 92$. This $(-1, 1)$ -FKTS is called a structurable algebra or for more general case, it is said to be a normal triality algebra ([1], [37]).

For subalgebras A_1 and A_2 of $\mathbf{O}$, if we use the notation $A = A_1 \otimes A_2$, $\dim A_1, \dim A_2$, then the Lie algebras obtained from these subalgebras are as follows:

$$L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \cong E_8, \quad L_0 \cong D_7 \oplus gl(1), \quad L_{-2} \oplus L_0 \oplus L_2 \cong D_8,$$

$\dim A_2 \backslash \dim A_1$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

For the case's E_8 , considering the extended Dynkin diagram, we have

$L_{-1} \oplus L_1 \cong L(U)/(L_{-2} \oplus L_0 \oplus L_2) = E_8/D_8$ with $\dim(L_{-1} \oplus L_1) = 128$;
 $\odot - \circ - \circ - \circ - \circ - \circ - \circ - \circ - \square, \square$ omitted $\cong D_8$, and $\odot$ is the highest root.



From these results, we obtain the bisymmetric space's series of tensor type, which is called a *tensor type bisymmetric space*,

$$(\heartsuit) \quad F_4/B_4 < E_6/(D_5 \oplus gl(1)) < E_7/(D_6 \oplus A_1) < E_8/D_8,$$

and the dimensions of bisymmetric spaces $(\heartsuit)$ of tensor type are

$$16, \quad 32, \quad 64, \quad 128.$$

With respect to extended Dynkin diagrams (symmetric spaces) of E_7 , E_6 , F_4 :

$\circ - \boxed{\circ} - \circ - \circ - \circ - \circ - \odot$, $\boxed{\circ}$ omitted $\cong D_6 \oplus A_1$, and $\odot$ is the highest root.

$$\begin{array}{c} | \\ \circ \\ \boxed{\circ} - \circ - \circ - \circ - \circ - \boxed{\circ}, \quad \boxed{\circ} \text{ omitted } \cong D_5 \oplus gl(1), \text{ and } \odot \text{ is the highest root.} \\ | \\ \circ \\ | \\ \odot \end{array}$$

$\odot - \circ - \circ - \circ = > \circ - \boxed{\circ}$, $\boxed{\circ}$ omitted $\cong B_4$, and $\odot$ is the highest root.

Remark Let $(U, < xyz >)$ be a (ε, δ) -FKTS. If $J \in \text{End } U$ satisfies $J^2 = -Id$ and $J < xyz > = < JxJyJz >$, then $(U, \{xyz\})$ is a $(-\varepsilon, \delta)$ -FKTS with respect the new product defined by $\{xyz\} = < xJyz >$.

6 A view point of geometric structures of $U(\varepsilon, \delta)$

6.1 A generalized curvature and torsion tensors associated with (ε, δ) -FKTS $U(\varepsilon, \delta)$

Let $L = L(U(\varepsilon, \delta)) = L(W, W) \oplus W$ be the Lie (super)algebra defined from a δ LTS $W = T(\varepsilon, \delta)$ and $L(W, W) = \text{Der } (T(\varepsilon, \delta), T(\varepsilon, \delta))$ as in section one, that is, the δ -LTS $W = T(\varepsilon, \delta) = L_{-1} \oplus L_1$ is induced from $L_{-1} = U(\varepsilon, \delta)$ (as L_{-1} has the structure of a (ε, δ) -FKTS) over the field Φ of $ch \ \Phi \neq 2, 3$.

We introduce a generalization of covariant derivative ∇ in differential geometry as follows; $\nabla : L \rightarrow \text{End } L$ defined by

$$\begin{aligned}\nabla_X Y &= [X, Y] = -\delta[Y, X], \\ \nabla_X [Y, Z] &= [Y Z X] = -\delta[Z Y X], \\ \nabla_{[X, Y]} Z &= -[X Y Z] = -\delta[Y X Z], \\ \nabla_{[X, Y]} [V, Z] &= [[V, Z][X, Y]] = -\delta[[X, Y][V, Z]], \\ \text{for any } X, Y, Z &\in W.\end{aligned}$$

Furthermore, a generalized curvature tensor defined by

$$C_\delta(X, Y) = \nabla_X \nabla_Y - \delta \nabla_Y \nabla_X - \nabla_{[X, Y]} \quad (12)$$

is identically zero, i.e., $C_\delta(X, Y) = 0$ in L , for any $X, Y \in W$. Indeed, we demonstrate the proof below.

First we calculate

$$\begin{aligned}C_\delta(X, Y)Z &= (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)Z - \nabla_{[X, Y]}Z \\ &= \nabla_X [Y, Z] - \delta \nabla_Y [X, Z] + [X Y Z] \\ &= [Y Z X] - \delta [X Z Y] + [X Y Z] \\ &= [Y Z X] + [Z X Y] + [X Y Z] = 0.\end{aligned}$$

Second, it follow

$$\begin{aligned}C_\delta(X, Y)[V, Z] &= (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)[V, Z] - \nabla_{[X, Y]}[V, Z] \\ &= [X, [V Z Y]] - \delta [Y, [V Z X]] + \delta [[X, Y], [V, Z]] \\ &= [X, L(V, Z)Y] - \delta [Y, L(V, Z)X] - L(V, Z)[X, Y] = 0 \\ \text{(by } [Y, L(V, Z)X] &= -\delta [L(V, Z)X, Y] \text{ and } [[X, Y], [V, Z]] = -\delta [[V, Z], [X, Y]]) \\ \text{for any } X, Y, Z, V &\in T(\varepsilon, \delta).\end{aligned}$$

However a generalized torsion tensor defined by

$$S_\delta(X, Y) = \nabla_X Y - \delta \nabla_Y X - [X, Y] \quad (13)$$

is not zero, since it gives $S_\delta(X, Y) = [X, Y] - \delta [Y, X] - [X, Y] = [X, Y]$.

To remember the J in section one, we have defined the Nijenhuis operator

$$N(X, Y) = [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY],$$

where for the case of $\varepsilon\delta = 1$, J is an almost complex structure on W , this concept (the case of $\delta = 1$) is appeared in ([38]). If we now set $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, or $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then we have $J^2 = -Id$, or $J^2 = Id$, respectively, hence these concept mean that there is a comlex or para comlex structures in the symmetric (super)space W , and furtermore, by means of the treatment of a wide choice for J , there is a structure of polarized, invariant, straight, twisted, etc. as well as ([5], [6]).

6.2 Metric structures associated with (ε, δ) -FKTS $U(\varepsilon, \delta)$

In this section, we will introduce a certain metric structure with respect to the (ε, δ) -FKTS $U(\varepsilon, \delta)$ over the field Φ of $ch \ \Phi \neq 2, 3$ as follows (hereby, it is said to be a *Killing form* of the triple system $U(\varepsilon, \delta)$);

$$\gamma(x, y) = \frac{1}{2}Trace[2(R(x, y) - \varepsilon R(y, x)) + \delta(\varepsilon L(x, y) - L(y, x))],$$

this bilinear form $\gamma(x, y)$ is a map: $U(\varepsilon, \delta) \times U(\varepsilon, \delta) \longrightarrow \Phi$ and denote by $L(x, y)z = \langle xyz \rangle$ and $R(x, y)z = \langle zxy \rangle$.

Next we can define the Killing form of $T(\varepsilon, \delta)$ associated with $U(\varepsilon, \delta)$;

$$\alpha(X, Y) = \frac{1}{2}Trace(R(X, Y) + \delta R(Y, X)),$$

and by using Eq.(10) in section one, we obtain $\alpha(X, Y) = \gamma(b_1, a_2) + \delta\gamma(b_2, a_1)$, where $X = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, Y = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in T(\varepsilon, \delta) = U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$.

Furthermore, for $\delta = \pm 1$, we have

$$\gamma(\langle xyz \rangle, w) + \delta\gamma(z, \langle yxw \rangle) = 0, \quad \gamma(\langle xyz \rangle, w) + \delta\gamma(x, \langle wzy \rangle) = 0,$$

$$\gamma(Dx, y) + \gamma(x, Dy) = 0, \text{ for all } D \in Der \ U(\varepsilon, \delta)$$

$$\alpha(X, Y) = \beta(Y, X) = \delta\beta(X, Y), \forall X, Y \in T(\varepsilon, \delta) \subset L(U(\varepsilon, \delta)),$$

where $\beta(X, Y) = strace \ ad \ X \ ad \ Y$, (the Killing form of Lie superalgebras). Indeed, from relations $gL(x, y)g^{-1} = L(gx, gy)$, $gR(x, y)g^{-1} = R(gx, gy)$, $[D, L(x, y)] = L(Dx, y) + L(x, Dy)$ and $[D, R(x, y)] = R(Dx, y) + R(x, Dy)$, it follows.

On the other hand, for special balanced cases of $U = U(\varepsilon, \delta)$, by straightforward calculations, we obtain the relation;

$$\gamma(x, y) = \frac{\delta}{2} \langle x, y \rangle \dim U + (1 - \delta) \langle x, y \rangle.$$

These results are appeared for a part study in ([14], [15], [18], [37]). However, we do not go the details.

6.3 A characterization of automorphisms of $U(\varepsilon, \delta)$

In this section, we will discuss with a certain automorphism of $T(\varepsilon, \delta)$ and Lie (super)algebras $L(U(\varepsilon, \delta))$ associated with an automorphism of (ε, δ) -Freudenthal-Kantor triple systems $U(\varepsilon, \delta)$ over the field Φ of $ch \ \Phi \neq 2, 3$.

Proposition 6.1 *Let $U(\varepsilon, \delta)(= U)$ be a (ε, δ) -Freudenthal-Kantor triple system, the Killing form $\gamma(x, y)$ of U and g be any automorphism of the triple system U . Then we have*

$$\gamma(x, y) = \gamma(gx, gy), \quad \text{for all } x, y \in U$$

Furthermore, $\gamma(S(x, y)z, w) + \gamma(z, S(x, y)w) = 0$, for all $x, y, z, w \in U$, where $S(x, y) := L(x, y) + \varepsilon L(y, x) \in \text{Der } U$.

Indeed, from $gR(x, y)g^{-1} = R(gx, gy)$ and $gL(x, y)g^{-1} = L(gx, gy)$, it follows.

It seems that this relation of $S(x, y)$ means a certain adjoint representation with respect to the $U(\varepsilon, \delta)$

Next, for any element $g \in \text{Epi}(U(\varepsilon, \delta))$, we define an endomorphism $\tilde{g}$ of $L(U(\varepsilon, \delta))$ as follows; $(\text{Der } T(\varepsilon, \delta)) \oplus T(\varepsilon, \delta) \longrightarrow$

$$\begin{pmatrix} 0 & \delta g \\ -\varepsilon g & 0 \end{pmatrix} (\text{Der } T(\varepsilon, \delta)) \begin{pmatrix} 0 & \delta g \\ -\varepsilon g & 0 \end{pmatrix}^{-1} \oplus \begin{pmatrix} 0 & \delta g \\ -\varepsilon g & 0 \end{pmatrix} (T(\varepsilon, \delta)).$$

In particular, if $g = Id$, then $\tilde{g} = \begin{pmatrix} 0 & \delta Id \\ -\varepsilon Id & 0 \end{pmatrix}$ is an automorphism of $L(U(\varepsilon, \delta))$.

Hence, under these notations, by straightforward calculations, we have the following.

Theorem 6.2 *The following are equivalent;*

- i) g is a linear transformation of $U(\varepsilon, \delta)$ satisfying $g \langle xyz \rangle = \langle gxgygz \rangle$,
- ii) $\tilde{g}$ is an automorphism of $T(\varepsilon, \delta)$,
- iii) $\tilde{g}$ is an automorphism of $L(U(\varepsilon, \delta))$.

Indeed, from $\tilde{g}[XYZ] = [(\tilde{g}X)(\tilde{g}Y)(\tilde{g}Z)]$, $\forall X, Y, Z \in T(\varepsilon, \delta)$, it follows.

Remark Note that for the Killing forms $\alpha(X, Y)$, $\beta(X, Y)$, we have $\beta(\tilde{g}X, \tilde{g}Y) = \beta(X, Y)$, and $\alpha(\tilde{g}X, \tilde{g}Y) = \alpha(X, Y)$, $\forall X, Y \in T(\varepsilon, \delta) \subset L(\varepsilon, \delta)$.

Concluding Remark

–In homage of Professor Susumu Okubo (1930-2015)–
he is a Wigner Medal Winner (2006) and a Nishina memorial prize (1976).

For relationships between triple systems and mathematical physics, we would like to refer S. Okubo's book ([60]) and ([23], [27], [39], [40], [41], [62]), and for a certain history in our subjects of nonassociative algebras, refer ([44]). In final comments of this paper, with respect to the nonassociative algebras (in particular, Lie admissible algebras), we note that there are a lot of important works due to Prof. R. Santilli for mathematical physics into Hadronic Journal's articles.

Appendix I (the case of $\varepsilon = 1$ and $\delta = 1$)

In the author's judgement, roughly speaking, there is a reason for studying triple systems as follows.

In this appendix, we note that there is a certain correspondence of our Lie algebra $L(U)$ and simple Lie algebra $\mathfrak{g}$ over an algebraically closed field Φ of characteristic zero with the highest root ρ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, Δ is the root system, Δ^+ is the positive root system, $\Delta_1 = \{\alpha \in \Delta^+ | \rho \neq \alpha, \rho - \alpha \in \Delta^+\}$, $\Delta_0 = \{\alpha \in \Delta^+ | \rho \neq \alpha, \rho - \alpha \notin \Delta^+\}$, That is, $\Delta^+ = \Delta_0 \cup \Delta_1 \cup \{\rho\}$. Then we have a decomposition of $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, $\dim \mathfrak{g}_\alpha = 1$.

Furthermore we choose certain basis $\{H_\rho, E_{\pm\alpha}, E_{\pm\beta}, E_{-\rho}, E_\rho | \alpha \in \Delta_1, \beta \in \Delta_0\}$ of subspaces $\sum \mathfrak{g}_\alpha$ ($\alpha \in \Delta$) and $H_\rho \in \mathfrak{h}$ satisfying the relations;
 $[E_{-\rho}, E_\rho] = H_\rho$, $[H_\rho, E_{-\rho}] = -2E_{-\rho}$, $[H_\rho, E_\rho] = 2E_\rho$, $[E_\rho, E_{-\alpha}] = E_{\rho-\alpha}$,
 $[E_{-\rho}, E_\alpha] = -E_{-\rho+\alpha}$, (if $\alpha \in \Delta_1$), $[E_\rho, E_{\pm\beta}] = 0$, (if $\beta \in \Delta_0$).
 Note that $\langle E_{-\rho}, E_\rho, H_\rho \rangle_{span} = sl(2)$, i.e., A_1 type.

Then we have a decomposition;

$\mathfrak{g} = \sum_{i=-2}^2 \mathfrak{g}_i$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, $[H_\rho, \mathfrak{g}_i] = i \mathfrak{g}_i, i = 0 \pm 1, \pm 2$
 where $\mathfrak{g}_{-2} = \mathfrak{g}_{-\rho} = \langle E_{-\rho} \rangle_{span}$, $\mathfrak{g}_2 = \mathfrak{g}_\rho = \langle E_\rho \rangle_{span}$, $\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$, $\mathfrak{g}_{-1} = \sum_{\alpha \in \Delta_1} \mathfrak{g}_{-\alpha}$, $\mathfrak{g}_1 = \sum_{\alpha \in \Delta_1} \mathfrak{g}_\alpha$, note that for $ad H_\rho$, this $\mathfrak{g}_i$ is the vector space of the eigen value i , because $ad H_\rho \mathfrak{g}_i = i \mathfrak{g}_i$ by $[H_\rho, E_\alpha] = E_\alpha$ (if $\alpha \in \Delta_1$) $[H_\rho, E_\beta] = 0$ (if $\beta \in \Delta_0$), $\mathfrak{g}_1 = \langle E_\alpha | \alpha \in \Delta_1 \rangle_{span}$, etc..

By means of notation being as above, a little long calculations, we can verify the following.

Theorem A.1. *Let $\mathfrak{g}$ be a simple Lie algebra and for the vector space $\mathfrak{g}_{-1}$ of $\mathfrak{g}$, then $U = \mathfrak{g}_{-1}$ is a balanced $(1, 1)$ -Freudenthal-Kantor triple system with respect to the triple product*

$L(x, y)z = \langle xyz \rangle = [[x, [y, E_\rho]], z]$, $\forall x, y, z \in \mathfrak{g}_{-1}$, equipped with $K(x, y) = -L(x, y) + L(y, x) = \langle x, y \rangle Id$, where the bilinear form $\langle x, y \rangle$ is defined by $[x, y] = \langle x, y \rangle E_{-\rho}$.

Furthermore, for the standard embedding Lie algebra $L(U)$ associated with $U = \mathfrak{g}_{-1}$ and the Lie triple system $T(1, 1) = U \oplus U$ (the case of $\varepsilon = 1, \delta = 1$) in section one, we have an isomorphism of Lie algebras as follows.

Theorem A.2. *For Lie algebras $L(U)$ and $\mathfrak{g}$ as above, we have $L(U) \simeq \mathfrak{g}$ as the Lie algebras isomorphism, by an isomorphism mapping ϕ , $\phi : L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$,*

$$D(T(1, 1), T(1, 1)) := \left\{ \begin{pmatrix} L(x, y) & Id \\ -Id & L(y, x) \end{pmatrix} \right\}_{span} \longrightarrow E_{-\rho} + ad([x, [y, E_\rho]]) + E_\rho,$$

$$T(1, 1) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U \right\}_{span} \longrightarrow x + [y, E_\rho].$$

where $\langle x, y \rangle E_{-\rho} = [x, y]$, $\forall x, y \in U$. Note that $\langle x, y \rangle E_\rho = [[x, E_\rho], [y, E_\rho]]$.

Appendix II (the case of $\varepsilon = -1$ and $\delta = 1$). In this appendix II, we describe another correspondence of $L(U)$ and $\mathfrak{g}$.

For simple Lie algebras $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ and the root system Δ , then we have $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

And we introduce notations as follows;

the highest root ρ ($\rho = 2\alpha_1 + \sum_{i \neq 1} n_i \alpha_i$), α_i is the simple roots,

$\Delta_0^* = \{\alpha \in \Delta | \alpha = \sum_{i \neq 1} m_i \alpha_i, m_i : \text{nonnegative integer}\}$,

$\Delta_1^* = \{\alpha \in \Delta | \alpha = \alpha_1 + \sum_{i \neq 1} k_i \alpha_i, k_i : \text{nonnegative integer}\}$,

$\Delta_2^* = \{\alpha \in \Delta \mid \alpha = 2\alpha_1 + \sum_{i \neq 1} l_i \alpha_i, l_i : \text{nonnegative integer}\}$.

Note that $\rho \in \Delta_2^*$, however, note that the highest root of A_{n-1} type's case with $(n^2 - 1)$ dimension is $\rho = \alpha_1 + \alpha_n + \sum_{i \neq 1, n} m_i \alpha_i$, where m_i is positive integers.

Then based on these notations, we have a decomposition;

$$\mathfrak{g} = \sum_{i=-2}^2 \mathfrak{g}_i, [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j},$$

where $\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_0^*} \mathfrak{g}_{\pm\alpha}$, $\mathfrak{g}_{\pm 2} = \sum_{\alpha \in \Delta_2^*} \mathfrak{g}_{\pm\alpha}$, $\mathfrak{g}_{\pm 1} = \sum_{\alpha \in \Delta_1^*} \mathfrak{g}_{\pm\alpha}$.

We consider an endomorphism τ of $\mathfrak{g}$ satisfying $\tau^2 = id$ and $\tau([X, Y]) = [\tau(X), \tau(Y)]$, $\forall X, Y \in \mathfrak{g}$, that is, $\tau : \mathfrak{g}_{-i} \rightarrow \mathfrak{g}_i$, by straightforward calculations, we obtain the following.

Theorem A.3. *The vector space $U = \mathfrak{g}_{-1}$ has a structure of $(-1, 1)$ -Freudenthal-Kantor triple system with respect to the triple product $L(x, y)z := \langle xyz \rangle = [[x, \tau(y)], z]$, $\forall x, y, z \in U$.*

Hence we can verify the following theorem.

Theorem A.4. *For Lie algebras $L(U)$ and $\mathfrak{g}$, we have*

$L(U) \simeq \mathfrak{g}$ as the Lie algebras isomorphism, by an isomorphism mapping ϕ , $\phi : L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$,

$$D(T(-1, 1), T(-1, 1)) := \left\{ \begin{pmatrix} L(x, y) & K(a, b) \\ K(c, d) & -L(y, x) \end{pmatrix} \right\}_{span} \longrightarrow$$

$$[a, b] + ad([x, \tau(y)]) + [\tau(c), \tau(d)],$$

$$T(-1, 1) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in U \right\}_{span} \longrightarrow x + \tau(y).$$

where $L(x, y)z = \langle xyz \rangle$, $K(x, y)z = \langle xzy \rangle - \langle yzx \rangle$, $\forall x, y, z \in U$.

With respect to an example of above τ of $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$, $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$, note that if $\theta : \mathfrak{h} \rightarrow -\mathfrak{h}$, ($\mathfrak{h} \in \mathfrak{h}$), and $E_{\alpha} \rightarrow -E_{-\alpha}$, ($E_{\alpha} \in \mathfrak{g}_{\alpha}$), then it satisfies $\theta^2 = id$, and $\theta([X, Y]) = [\theta(X), \theta(Y)]$, $\forall X, Y \in \mathfrak{g}$.

Appendix III (the case of $\varepsilon = -1$ and $\delta = -1$)

In this appendix, we note that there is a certain correspondence of our Lie superalgebra $L(U)$ and 5-graded simple basic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over an algebraically closed field Φ of characteristic zero with nondegenerate Killing form, (for example, $A(m, n)$, $B(m, n)$, $D(m, n)$, $G(3)$, $F(4)$), now let the highest root be ρ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, Δ be the root system, Δ^+ be the positive root system, $\Delta_1 = \{\alpha \in \Delta^+ \mid \rho \neq$

$\alpha, \rho - \alpha \in \Delta^+\}$, $\Delta_0 = \{\alpha \in \Delta^+ | \rho \neq \alpha, \rho - \alpha \notin \Delta^+\}$, That is, $\Delta^+ = \Delta_0 \cup \Delta_1 \cup \{\rho\}$. Then we have a decomposition of $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, $\dim \mathfrak{g}_\alpha = 1$.

Furthermore we choose certain basis $\{H_\rho, E_{\pm\alpha}, E_{\pm\beta}, E_{-\rho}, E_\rho | \alpha \in \Delta_1, \beta \in \Delta_0\}$ of subspaces $\sum \mathfrak{g}_\alpha$ ($\alpha \in \Delta$) and $H_\rho \in \mathfrak{h}$ satisfying the relations;
 $[E_{-\rho}, E_\rho] = H_\rho, [H_\rho, E_{-\rho}] = -2E_{-\rho}, [H_\rho, E_\rho] = 2E_\rho, [E_\rho, E_{-\alpha}] = E_{\rho-\alpha},$
 $[E_{-\rho}, E_\alpha] = -E_{-\rho+\alpha}$, (if $\alpha \in \Delta_1$), $[E_\rho, E_{\pm\beta}] = 0$, (if $\beta \in \Delta_0$).
 Note that $\langle E_{-\rho}, E_\rho, H_\rho \rangle_{span} = sl(2)$, i.e., A_1 type.

Then we have a decomposition;

$$\mathfrak{g} = \sum_{i=-2}^2 \mathfrak{g}_i, [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, [H_\rho, \mathfrak{g}_i] = i \mathfrak{g}_i, i = 0 \pm 1, \pm 2$$

where $\mathfrak{g}_{-2} = \mathfrak{g}_{-\rho} = \langle E_{-\rho} \rangle_{span}$, $\mathfrak{g}_2 = \mathfrak{g}_\rho = \langle E_\rho \rangle_{span}$, $\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$, $\mathfrak{g}_{-1} = \sum_{\alpha \in \Delta_1} \mathfrak{g}_{-\alpha}$, $\mathfrak{g}_1 = \sum_{\alpha \in \Delta_1} \mathfrak{g}_\alpha$, note that for $ad H_\rho$, this $\mathfrak{g}_i$ is the vector space of the eigen value i , because $ad H_\rho \mathfrak{g}_i = i \mathfrak{g}_i$ by $[H_\rho, E_\alpha] = E_\alpha$ (if $\alpha \in \Delta_1$) $[H_\rho, E_\beta] = 0$ (if $\beta \in \Delta_0$), $\mathfrak{g}_1 = \langle E_\alpha | \alpha \in \Delta_1 \rangle_{span}$, and $\mathfrak{g}_0 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$, $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

By means of notation being as above, a little long calculations, we can verify the following.

Theorem A.5. *Let $\mathfrak{g}$ be a simple Lie superalgebra as above and for the vector space $\mathfrak{g}_{-1}$ of $\mathfrak{g}$, then $U = \mathfrak{g}_{-1}$ is a balanced $(-1, -1)$ -Freudenthal-Kantor triple system with respect to the triple product*

$$L(x, y)z = \langle xyz \rangle = [[x, [y, E_\rho]], z], \quad \forall x, y, z \in \mathfrak{g}_{-1}, \text{ equipped with } K(x, y) = L(x, y) + L(y, x) = \langle x, y \rangle, \text{ where the bilinear form } \langle x, y \rangle \text{ is defined by } [x, y] = -\langle x, y \rangle E_{-\rho}.$$

Furthermore, for the standard embedding Lie superalgebra $L(U)$ associated with $U = \mathfrak{g}_{-1}$ and the anti-Lie triple system $T(-1, -1) = U \oplus U$ (the case of $\varepsilon = -1, \delta = -1$) in section one, we have an isomorphism of Lie algebras as follows.

Theorem A.6. *For Lie superalgebras $L(U)$ and $\mathfrak{g}$ as above, we have $L(U) \simeq \mathfrak{g}$ as the Lie superalgebras isomorphism, by an isomorphism mapping ϕ ,*

$$\phi : L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

$$D(T(-1, -1), T(-1, -1)) := \left\{ \begin{pmatrix} L(x, y) & -Id \\ Id & -L(y, x) \end{pmatrix} \right\}_{span} \longrightarrow$$

$$E_{-\rho} + ad([x, [y, E_\rho]]) + E_\rho,$$

$$T(-1, -1) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U \right\}_{span} \longrightarrow x - [y, E_\rho].$$

where $\langle x, y \rangle E_{-\rho} = -[x, y]$, $\forall x, y \in U$. Note that $\langle x, y \rangle E_\rho = -[[x, E_\rho], [y, E_\rho]]$.

Proof. Indeed, it is enough to show that

$$\phi\left(\left[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}\right]\right) = \left(\left[\phi\begin{pmatrix} a \\ b \end{pmatrix} \phi\begin{pmatrix} c \\ d \end{pmatrix} \phi\begin{pmatrix} e \\ f \end{pmatrix}\right]\right) = \left(\left[\left[\phi\begin{pmatrix} a \\ b \end{pmatrix}, \phi\begin{pmatrix} c \\ d \end{pmatrix}\right], \phi\begin{pmatrix} e \\ f \end{pmatrix}\right]\right).$$

This is a straight forward calculations, but we omit it.

Appendix IV (the case of $\varepsilon = 1$ and $\delta = -1$). In this appendix IV, we describe another correspondence of $L(U)$ and $\mathfrak{g}$ as in appendix III.

For simple Lie superalgebras $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ and the root system Δ , then we have $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

And we introduce notations as follows;

the highest root ρ ($\rho = 2\alpha_1 + \sum_{i \neq 1} n_i \alpha_i$), α_i is the simple roots,

$$\Delta_0^* = \{\alpha \in \Delta \mid \alpha = \sum_{i \neq 1} m_i \alpha_i, m_i : \text{nonnegative integer}\},$$

$$\Delta_1^* = \{\alpha \in \Delta \mid \alpha = \alpha_1 + \sum_{i \neq 1} k_i \alpha_i, k_i : \text{nonnegative integer}\},$$

$$\Delta_2^* = \{\alpha \in \Delta \mid \alpha = 2\alpha_1 + \sum_{i \neq 1} l_i \alpha_i, l_i : \text{nonnegative integer}\}.$$

Note that $\rho \in \Delta_2^*$, however, note that for $A(n-1, 1) = sl(n|2)$ type's case with $(n^2 + 4n + 3)$ dimension, the highest root is $\rho = \alpha_1 + \alpha_n + \sum_{i \neq 1, n} m_i \alpha_i$, m_i is nonnegative integers.

Then based on these notations, we have a decomposition;

$$\mathfrak{g} = \sum_{i=-2}^2 \mathfrak{g}_i, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j},$$

$$\text{where } \mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_0^*} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{g}_{\pm 2} = \sum_{\alpha \in \Delta_2^*} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{g}_{\pm 1} = \sum_{\alpha \in \Delta_1^*} \mathfrak{g}_{\pm\alpha}.$$

We consider an endomorphism τ of $\mathfrak{g}$ satisfying $\tau^2 = id$ and $\tau([X, Y]) = [\tau(X), \tau(Y)]$, $\forall X, Y \in \mathfrak{g}$, that is, $\tau : \mathfrak{g}_{-i} \rightarrow \mathfrak{g}_i$, by straightforward calculations, we obtain the following.

Theorem A.7. *The vector space $U = \mathfrak{g}_{-1}$ has a structure of $(1, -1)$ -Freudenthal-Kantor triple system with respect to the triple product $L(x, y)z := \langle xyz \rangle = [[x, \tau(y)], z]$, $\forall x, y, z \in U$.*

Hence we can verify the following theorem.

Theorem A.8. *For Lie superalgebras $L(U)$ and $\mathfrak{g}$ as in Theorem A.7, we have*

$L(U) \simeq \mathfrak{g}$ as the Lie superalgebras isomorphism, by an isomorphism mapping ϕ ,

$$\phi: L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

$$D(T(1, -1), T(1, -1)) := \left\{ \begin{pmatrix} L(x, y) & -K(a, b) \\ -K(c, d) & L(y, x) \end{pmatrix} \right\}_{\text{span}} \longrightarrow$$

$$[a, b] + ad([x, \tau(y)]) + [\tau(c), \tau(d)],$$

$$T(1, -1) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U \right\}_{\text{span}} \longrightarrow x + \tau(y).$$

where $L(x, y)z = \langle xyz \rangle$, $K(x, y)z = \langle xzy \rangle + \langle yzx \rangle$, $\forall x, y, z \in U$.

Remark Let $(U, \langle xyz \rangle)$ be a (ε, δ) -FKTS. If $J \in \text{End } U$ satisfied $J^2 = Id$ and $J \langle xyz \rangle = \langle JxJyJz \rangle$, then $(U, \{xyz\})$ is a $(-\varepsilon, \delta)$ -FKTS with respect to the product $\{xyz\} = \langle xJyz \rangle$.

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**A VARIATIONAL FORMULATION FOR MODELING A
PROTIUM HYDROGEN MOLECULAR IONIZATION**

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Abstract

This article develops a variational formulation for modeling a protium hydrogen molecular ionization obtained through a high temperature scalar field and an appropriate electric one action. The results are based on standard tools of calculus of variations and optimization theory. Finally, we highlight the context here addressed is essentially an Euler-Bernoullian one and includes the establishment of a new approximate Bernoulli-interacting-gas type equation.

AMS Classification 35Q70, 35Q40, 76N15

Keywords. Hydrogen molecular system, ionization, strong variational formulations, Lagrange multipliers, optimization.

1 Introduction

The first part of this article develops a variational formulation for modeling the ionization of a gaseous hydrogen molecular system, through a high temperature field and an appropriate electric field. The context here developed is an Euler-Bernoullian one.

Such results are based on tools of calculus of variations and related Lagrange multipliers approach.

It is worth emphasizing we have obtained a new system of partial differential equations for modeling a compressible case in fluid mechanics, similar to those found in the contemporary literature however, with a new Bernoullian-interacting-gas equation replacing a standard ideal gas equation and related interacting one, in such a concerning model. A related similar result for a simpler case may be found in [1].

We also highlight the two main fields for such a system are a position one and a velocity one, which we specify below in the next lines.

As standard references in theoretical fluid mechanics we would cite [2, 3, 4, 5, 6]. For related numerical methods we would mention [7, 8, 9, 10]. Concerning similar models, we would cite [11, 12, 13, 14, 15, 16].

Finally, details on the Sobolev spaces involved may be found in [17].

2 A note on the first Maxwell equation of electromagnetism

Firstly, we highlight the main results in this section have been previously presented in similar form in the Preprint [18].

Let $\Omega_1 \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega_1$.

Suppose $\mathbf{E} : \Omega_1 \rightarrow \mathbb{R}^3$ is an electric field of resulting from a punctual charge q localized at $(0, 0, 0) \in \Omega_1$.

Let $\Omega \subset \Omega_1$ be also an open, bounded and connected set with a regular (C^1 class) boundary denoted by $S = \partial\Omega$, such that $(0, 0, 0) \in \Omega$, where, in polar coordinates $(r, \theta, \phi) \in \mathbb{R}^3$, we have

$$S = \{\mathbf{r}(\theta, \phi) : 0 \leq \theta \leq \pi \text{ and } 0 \leq \phi \leq 2\pi\},$$

and where,

$$\mathbf{r}(\theta, \phi) = X(\theta, \phi)\mathbf{i} + Y(\theta, \phi)\mathbf{j} + Z(\theta, \phi)\mathbf{k}.$$

and

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1) \in \mathbb{R}^3.$$

Observe that denoting

$$\mathbf{e}_r = \sin(\theta) \cos(\phi) \mathbf{i} + \sin(\theta) \sin(\phi) \mathbf{j} + \cos(\theta) \mathbf{k},$$

we have

$$\mathbf{E} = \hat{K} q \frac{1}{r^2} \mathbf{e}_r,$$

for an appropriate real constant $\hat{K}$.

We highlight the integral

$$I_n = \int_S \mathbf{E} \cdot \mathbf{n} \, dS = c_5$$

where such a real constant c_5 does not depend on the C^1 class function

$$\mathbf{r} : [0, \pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3.$$

Here $\mathbf{n}$ denotes the normal outward field to S .

In this section, we develop such an integral I_n , in details.

Denoting

$$R(\theta, \phi) = \|\mathbf{r}(\theta, \phi)\|_{\mathbb{R}^3},$$

we have

$$\mathbf{r}(\theta, \phi) = R(\theta, \phi) \mathbf{e}_r.$$

Moreover, we recall that

$$dS = \|\mathbf{r}_\theta \times \mathbf{r}_\phi\|_{\mathbb{R}^3} \, d\theta d\phi,$$

where

$$N = \mathbf{r}_\theta \times \mathbf{r}_\phi$$

is such that

$$\mathbf{n} = \frac{N}{\|N\|_{\mathbb{R}^3}}.$$

Thus

$$\int_S \mathbf{E} \cdot \mathbf{n} \, dS = \hat{K} q \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{1}{R(\theta, \phi)^2} \mathbf{e}_r \cdot \frac{N}{\|N\|_{\mathbb{R}^3}} \|N\|_{\mathbb{R}^3} \, d\theta d\phi.$$

In summary,

$$\int_S \mathbf{E} \cdot \mathbf{n} \, dS = \hat{K} q \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{1}{R(\theta, \phi)^2} \mathbf{e}_r \cdot N \, d\theta \, d\phi.$$

Observe that

$$\begin{aligned}
 N &= (R(\theta, \phi) \mathbf{e}_r)_\theta \times (R(\theta, \phi) \mathbf{e}_r)_\phi \\
 &= (R_\theta \mathbf{e}_r + R(\mathbf{e}_r)_\theta) \times (R_\phi \mathbf{e}_r + R(\mathbf{e}_r)_\phi) \\
 &= R_\theta R_\phi \mathbf{e}_r \times \mathbf{e}_r + R_\theta \mathbf{e}_r \times R(\mathbf{e}_r)_\phi \\
 &\quad + R(\mathbf{e}_r)_\theta \times (R_\phi \mathbf{e}_r) + R^2(\mathbf{e}_r)_\theta \times (\mathbf{e}_r)_\phi.
 \end{aligned} \tag{1}$$

Consequently, from such results, we obtain

$$\mathbf{e}_r \cdot N = R^2 \mathbf{e}_r \cdot (\mathbf{e}_r)_\theta \times (\mathbf{e}_r)_\phi.$$

Thus, we have got

$$\begin{aligned}
 &\int_S \mathbf{E} \cdot \mathbf{n} \, dS \\
 &= \hat{K} q \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{1}{R(\theta, \phi)^2} \mathbf{e}_r \cdot N \, d\theta \, d\phi \\
 &= \hat{K} q \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{1}{R(\theta, \phi)^2} R(\theta, \phi)^2 \mathbf{e}_r \cdot (\mathbf{e}_r)_\theta \times (\mathbf{e}_r)_\phi \, d\theta \, d\phi \\
 &= \hat{K} q \, 4\pi \\
 &\equiv c_5.
 \end{aligned} \tag{2}$$

Summarizing, we have got

$$\int_S \mathbf{E} \cdot \mathbf{n} \, dS = c_5.$$

where such a real constant does not depend on function $\mathbf{r} : [0, \pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3$.

Consider now a charge q_0 localized at the center of a sphere Ω_2 of radius $R > 0$ and boundary $S_2 = \partial\Omega_2$.

The electric field on the sphere surface generated by q_0 is given by

$$\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0} \frac{q_0}{R^2} \mathbf{n}_2,$$

where $\mathbf{n}_2$ is the normal outward field to S_2 .

Clearly

$$\int_{S_2} \mathbf{E}_2 \cdot \mathbf{n}_2 \, dS_2 = \frac{1}{4\pi\epsilon_0} \frac{q_0}{R^2} (4\pi R^2) = \frac{q_0}{\epsilon_0}.$$

Consider again the set Ω but now with a charge q_0 localized at a point $\mathbf{x}$ inside the interior of Ω , which is denoted by Ω^0 .

At first the electric field $\mathbf{E}$ generated by q_0 is not of C^1 class on Ω .
However, there exists $R > 0$ such that

$$B_R(\mathbf{x}) \subset \Omega = \Omega^0.$$

Define $\Omega_3 = \Omega \setminus B_R(\mathbf{x})$.

Therefore, $\mathbf{E}$ is of C^1 class on Ω_3 .

Denoting the boundary of Ω_3 by $S_3 = \partial\Omega \cup \partial B_R(\mathbf{x})$, from the previous results and denoting again $S = \partial\Omega$, we may infer that

$$\int_{S_3} \mathbf{E} \cdot \mathbf{n} dS_3 = \int_S \mathbf{E} \cdot \mathbf{n} dS - \int_{\partial B_R(\mathbf{x})} \mathbf{E} \cdot \mathbf{n} dS_2 = c_5 - c_5 = 0,$$

for a not relabeled real constant c_5 , so that

$$\begin{aligned} \int_{S_3} \mathbf{E} \cdot \mathbf{n} dS_3 &= \int_S \mathbf{E} \cdot \mathbf{n} dS - \int_{\partial B_R(\mathbf{x})} \mathbf{E} \cdot \mathbf{n} dS_2 \\ &= \int_S \mathbf{E} \cdot \mathbf{n} dS - \frac{q_0}{\varepsilon_0} \\ &= 0. \end{aligned} \tag{3}$$

Therefore, we have got

$$\int_S \mathbf{E} \cdot \mathbf{n} dS = \frac{q_0}{\varepsilon_0}.$$

Assume now on Ω we have a density of charges $\rho(\mathbf{x})$.

For a small volume $\Delta\tilde{V}$ consider a punctual charge q_0 localized in $\mathbf{x} \in \Omega$ such that

$$q_0 \approx \rho(\mathbf{x})\Delta\tilde{V}.$$

Denoting by $\Delta\mathbf{E}$ the electric field generated by q_0 , from the previous results we may infer that

$$\int_S \Delta\mathbf{E} \cdot \mathbf{n} dS = \frac{q_0}{\varepsilon_0} \approx \frac{\rho(\mathbf{x})\Delta\tilde{V}}{\varepsilon_0}.$$

Such an equation in its differential form, stands for:

$$\int_S d\mathbf{E} \cdot \mathbf{n} dS = \frac{\rho(\mathbf{x}) d\tilde{V}}{\varepsilon_0}.$$

Integrating in Ω we may obtain

$$\begin{aligned}\int_S \mathbf{E} \cdot \mathbf{n} dS &= \int_S \int_{\Omega} d\mathbf{E} \cdot \mathbf{n} dS \\ &= \int_{\Omega} \frac{\rho(\mathbf{x})}{\varepsilon_0} d\tilde{V},\end{aligned}\tag{4}$$

so that

$$\int_S \mathbf{E} \cdot \mathbf{n} dS = \int_{\Omega} \frac{\rho(\mathbf{x})}{\varepsilon_0} d\tilde{V}.$$

From this and the Divergence Theorem, we have

$$\int_S \mathbf{E} \cdot \mathbf{n} dS = \int_{\Omega} \operatorname{div} \mathbf{E} d\tilde{V} = \int_{\Omega} \frac{\rho(\mathbf{x})}{\varepsilon_0} d\tilde{V}.$$

Summarizing, we have got

$$\int_{\Omega} \operatorname{div} \mathbf{E} d\tilde{V} = \int_{\Omega} \frac{\rho(\mathbf{x})}{\varepsilon_0} d\tilde{V}.$$

This is the integral form of the first Maxwell equation of electromagnetism.

For this last equation, the set $\Omega \subset \Omega_1$ is rather arbitrary so that for Ω as a ball of small radius $r > 0$ with center at a point $\mathbf{x} \in \Omega_1$, from the Mean Value Theorem for integrals and letting $r \rightarrow 0^+$, we obtain

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0}, \text{ in } \Omega_1.$$

This last equation stands for the differential form of the first Maxwell equation of electromagnetism.

Remark 2.1. *Summarizing, in this section we have formally obtained a mathematical deduction of the first Maxwell equation of electromagnetism.*

Remark 2.2. *Suppose beyond the electric field $\mathbf{E}$ generated by a charge density ρ , we have an external one associated with an external electric C^1 class potential V .*

Hence, the total electric field, here denoted by $\mathbf{E}_{total}$, is given by

$$\mathbf{E}_{total} = \mathbf{E} - \nabla V.$$

Hence

$$\mathbf{E} = \mathbf{E}_{total} + \nabla V,$$

so that from the previous results obtained, we have

$$\operatorname{div} \mathbf{E} = \operatorname{div} (\mathbf{E}_{total} + \nabla V) = \frac{\rho}{\varepsilon_0}.$$

Finally, considering other unit systems, we may find in the literature also the following expression for this first Maxwell equation:

$$\operatorname{div} \mathbf{E} = 4\pi\rho.$$

3 About a first variational formulation for compressible fluid mechanics including electromagnetic fields

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and simply connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Generically, we consider a fluid motion initially modeled as a particle system one, through the field of position $\mathbf{r} : \Omega \times [0, T] \rightarrow \mathbb{R}^3$ and field of velocities $\mathbf{v}$ where $\mathbf{v}(\mathbf{r}(x, t), t)$ stands for the velocity at the position $\mathbf{r}(x, t)$ and time $t \in [0, T]$.

Consider a functional J which represents a variational formulation for such a motion, where

$$\begin{aligned} J(\mathbf{r}, \mathbf{v}, \rho) = & \frac{1}{2} \int_0^T \int_{\Omega} \rho(\mathbf{r}(x, t), t) \mathbf{v}(\mathbf{r}(x, t), t) \cdot \mathbf{v}(\mathbf{r}(x, t), t) \, dxdt \\ & + \int_0^T \int_{\Omega} \lambda \left(\mathbf{v}(\mathbf{r}(x, t), t) - \frac{\partial \mathbf{r}(x, t)}{\partial t} \right) \, dxdt \\ & - \int_0^T \int_{\Omega} \hat{P} \left(\frac{d\rho}{dt} + \rho \operatorname{div} (\mathbf{v}) \right) \, dxdt \\ & - \int_0^T \int_{\Omega} \rho \mathbf{g} \cdot \mathbf{r} \, dxdt \end{aligned} \quad (5)$$

Here we have generically denoted

$$\rho = \rho(\mathbf{r}(x, t), t),$$

$$\mathbf{v} = \mathbf{v}(\mathbf{r}(x, t), t),$$

and

$$\mathbf{r} = \mathbf{r}(x, t).$$

In a second step, for an electronic field, we include electric and magnetic fields

and associated potentials in such a variational formulation, obtaining

$$\begin{aligned}
 J_1(\mathbf{r}, \mathbf{v}, \rho, \mathbf{A}, V) &= J(\mathbf{r}, \mathbf{v}, \rho) - K_1 \int_0^T \int_{\Omega} \rho \mathbf{A} \cdot \frac{\partial \mathbf{r}(x, t)}{\partial t} dx dt \\
 &\quad - K_1 \int_0^T \int_{\Omega} V(\mathbf{r}) \rho dx dt + \frac{1}{8\pi} \| \text{Curl } \mathbf{A} - \mathbf{B}_0 \|_{0,2,\Omega \times [0,T]}^2 \\
 &\quad - \frac{1}{8\pi} \left\| \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right\|_{0,2,\Omega \times [0,T]}^2 \\
 &\quad - \int_0^T \int_{\Omega} \rho \mathbf{B}(x, t) \times \frac{\partial \mathbf{r}(x, t)}{\partial t} \cdot (\mathbf{r}(\varepsilon t, x) - (\mathbf{r}_0)(x)) dx dt \\
 &= \frac{1}{2} \int_0^T \int_{\Omega} \rho(\mathbf{r}(x, t), t) \mathbf{v}(\mathbf{r}(x, t), t) \cdot \mathbf{v}(\mathbf{r}(x, t), t) dx dt \\
 &\quad + \int_0^T \int_{\Omega} \lambda \left(\mathbf{v}(\mathbf{r}(x, t), t) - \frac{\partial \mathbf{r}(x, t)}{\partial t} \right) dx dt \\
 &\quad - \int_0^T \int_{\Omega} \hat{P} \left(\frac{d\rho}{dt} + \rho \text{div}(\mathbf{v}) \right) dx dt \\
 &\quad - \int_0^T \int_{\Omega} \rho \mathbf{g} \cdot \mathbf{r} dx dt \\
 &\quad - K_1 \int_0^T \int_{\Omega} \rho \mathbf{A} \cdot \frac{\partial \mathbf{r}(x, t)}{\partial t} dx dt \\
 &\quad - K_1 \int_0^T \int_{\Omega} V(\mathbf{r}) \rho dx dt + \frac{1}{8\pi} \| \text{Curl } \mathbf{A} - \mathbf{B}_0 \|_{0,2,\Omega \times [0,T]}^2 \\
 &\quad - \frac{1}{8\pi} \left\| \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right\|_{0,2,\Omega \times [0,T]}^2 \\
 &\quad - \int_0^T \int_{\Omega} \rho \mathbf{B}(x, t) \times \frac{\partial \mathbf{r}(x, t)}{\partial t} \cdot (\mathbf{r}(\varepsilon t, x) - (\mathbf{r}_0)(x)) dx dt \quad (6)
 \end{aligned}$$

where,

$$\mathbf{B} = \mathbf{B}_0 - \text{Curl } \mathbf{A}$$

is the total magnetic field, $\mathbf{B}_0$ is an external magnetic field and $\mathbf{A} = (A_1, A_2, A_3)$ is the magnetic potential,

$$V(\mathbf{r})$$

is the electric potential and

$$\mathbf{E} = \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t},$$

is the electric field.

From the variation of J_1 in $\mathbf{v}$, we have

$$\rho \mathbf{v} + \lambda + \nabla(\rho \hat{P}) = \mathbf{0},$$

$\in \Omega \times [0, T]$.

From the variation of P in ρ , we obtain

$$\frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \frac{d\hat{P}}{dt} - \hat{P} \operatorname{div}(\mathbf{v}) + K_1 \mathbf{A} \cdot \frac{\partial \mathbf{r}}{\partial t} + K_1 V - \mathbf{g} \cdot \mathbf{r} = 0,$$

$\in \Omega \times [0, T]$.

From the variation of J_1 in $\mathbf{r}$, we obtain

$$\begin{aligned} \frac{dJ_1}{d\mathbf{r}} &= \frac{\partial J_1}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \\ &\quad + \frac{\partial J_1}{\partial \rho} \frac{\partial \rho}{\partial \mathbf{r}} \\ &\quad + \frac{\partial J_1}{\partial \mathbf{A}} \frac{\partial \mathbf{A}}{\partial \mathbf{r}} \\ &\quad + \frac{\partial J_1}{\partial V} \frac{\partial V}{\partial \mathbf{r}} + \frac{\partial J_1}{\partial \mathbf{r}} \\ &= \frac{\partial J_1}{\partial \mathbf{r}} \\ &= \mathbf{0}. \end{aligned} \tag{7}$$

Such a Gâteaux differential

$$\frac{dJ_1}{d\mathbf{r}}$$

is related to the following variation of J_1 in $\mathbf{r}$, which lead us to the correct Euler-Lagrange equation,

$$\begin{aligned} \left\langle \frac{dJ_1}{d\mathbf{r}}, \varphi \right\rangle_{L^2} &= \hat{\delta}(J_1)_{\mathbf{r}}(\mathbf{r}, \mathbf{r}(\varepsilon t, x), \mathbf{v}, \rho, \mathbf{A}, V; \varphi) \\ &= \lim_{h \rightarrow 0} [J_1(\mathbf{r} + h\varphi, \mathbf{r}(\varepsilon t, x) + h\varphi, \mathbf{v}, \rho, \mathbf{A}, V) \\ &\quad - J_1(\mathbf{r}, \mathbf{r}(\varepsilon t, x), \mathbf{v}, \rho, \mathbf{A}, V)] / h \\ &= \mathbf{0}, \quad \forall \varphi \in C_c^\infty(\Omega \times [0, T]; \mathbb{R}^3). \end{aligned} \tag{8}$$

The corresponding Euler-Lagrange equation stands for,

$$\frac{\partial \lambda}{\partial t} - \rho \mathbf{g} + K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} - K_1 \rho \mathbf{B} \times \mathbf{v} + \mathcal{O}(\varepsilon) = \mathbf{0},$$

in $\Omega \times [0, T]$.

From the variation of J_1 in $\hat{P}$, we have

$$\frac{d\rho}{dt} + \rho \operatorname{div}(\mathbf{v}) = 0,$$

Consequently, from such previous results, letting $\varepsilon \rightarrow 0^+$, we obtain

$$\begin{aligned} \mathbf{0} &= \frac{\partial \lambda}{\partial t} - \rho \mathbf{g} + K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} - K_1 \rho \mathbf{B} \times \mathbf{v} \\ &= -\frac{d(\rho \mathbf{v})}{dt} - \frac{d\nabla(\rho \hat{P})}{dt} \\ &\quad - \rho \mathbf{g} + K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} \\ &\quad - K_1 \rho \mathbf{B} \times \mathbf{v}. \end{aligned} \tag{9}$$

Now defining $P \equiv \frac{d(\rho \hat{P})}{dt}$, we obtain

$$\begin{aligned} &\frac{d(\rho \mathbf{v})}{dt} + \nabla P \\ &\quad + \rho \mathbf{g} - K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} + K_1 \rho \mathbf{v} \times \mathbf{B} \\ &= \mathbf{0}, \end{aligned} \tag{10}$$

in $\Omega \times [0, T]$.

Observe that

$$\begin{aligned} P &= \frac{d(\rho \hat{P})}{dt} \\ &= \frac{d\rho}{dt} \hat{P} + \rho \frac{d\hat{P}}{dt}. \end{aligned} \tag{11}$$

Thus,

$$\frac{P}{\rho} = \frac{d\rho}{dt} \frac{\hat{P}}{\rho} + \frac{d\hat{P}}{dt},$$

so that

$$\begin{aligned} \frac{P}{\rho} &= \frac{d\rho}{dt} \frac{\hat{P}}{\rho} - \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + K_1 \mathbf{A} \cdot \frac{\partial \mathbf{r}}{\partial t} - K_1 V + \mathbf{g} \cdot \mathbf{r} + \hat{P} \operatorname{div}(\mathbf{v}) \\ &= \left(\frac{d\rho}{dt} + \rho \operatorname{div}(\mathbf{v}) \right) \frac{\hat{P}}{\rho} - \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + K_1 \mathbf{A} \cdot \frac{\partial \mathbf{r}}{\partial t} - K_1 V + \mathbf{g} \cdot \mathbf{r} \\ &= -\frac{1}{2} \mathbf{v} \cdot \mathbf{v} + K_1 \mathbf{A} \cdot \frac{\partial \mathbf{r}}{\partial t} - K_1 V + \mathbf{g} \cdot \mathbf{r}. \end{aligned} \tag{12}$$

Therefore, we have obtained the Bernoulli type equation

$$\frac{P}{\rho} + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} - \mathbf{g} \cdot \mathbf{r} = K_1 \mathbf{A} \cdot \frac{\partial \mathbf{r}}{\partial t} - K_1 V.$$

Remark 3.1. Observe that the variation of J in V stands for

$$K_1\rho - \frac{1}{4\pi} \operatorname{div} \left(\nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

Hence, recalling that

$$\mathbf{E} = \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t},$$

we have obtained

$$\operatorname{div} \mathbf{E} = 4\pi K_1\rho.$$

Considering the statements in remark 2.2, we have that such an electric field $\mathbf{E}$ is the one exclusively generated by the density ρ so that, also from such a remark, we have

$$\mathbf{E} = \mathbf{E}_{total} + \nabla V = \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.$$

Thus, from such results, the total electric field stands for

$$\mathbf{E}_{total} = \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.$$

Hence, for a time independent ρ , the associated total electric field force density stands for

$$\mathbf{f}_e = -K_1 \frac{\rho}{c} \frac{\partial \mathbf{A}}{\partial t},$$

where, in atomic units, we have $c = 1$, so that

$$\mathbf{f}_e = -K_1 \rho \frac{\partial \mathbf{A}}{\partial t}.$$

Moreover, the density force associated with the magnetic field is given by

$$\mathbf{f}_m = K_1 \rho \mathbf{v} \times \mathbf{B},$$

so that

$$\mathbf{f}_e + \mathbf{f}_m = -K_1 \frac{\rho}{c} \frac{\partial \mathbf{A}}{\partial t} + K_1 \rho \mathbf{v} \times \mathbf{B}.$$

Indeed for a time dependent density, we have found the following Euler equation as an equilibrium one,

$$\begin{aligned} & \frac{d(\rho \mathbf{v})}{dt} + \nabla P \\ & + \rho \mathbf{g} - K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} + K_1 \rho \mathbf{v} \times \mathbf{B} \\ & = \mathbf{0}, \end{aligned} \tag{13}$$

in $\Omega \times [0, T]$.

Considering the statements in remark 2.2, such an equation agrees with the previous theoretical results obtained in the contemporary literature.

Remark 3.2. We are assuming ε_0 is a real constant. In a more general context we may consider a variable

$$\varepsilon_0 = \varepsilon_0(\rho(\mathbf{r}(x, t)), \mathbf{r}(x, t)).$$

In such a case, it is necessary to replace the functional part in atomic units

$$J_7 = -\frac{1}{2} \left\| \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right\|_{0,2}^2 = -\frac{1}{2} \|\mathbf{E}\|_{0,2}^2,$$

by

$$\hat{J}_7 = -\frac{1}{2} \int_0^T \int_{\Omega} \varepsilon_0(\rho(\mathbf{r}(x, t)), \mathbf{r}) \mathbf{E} \cdot \mathbf{E} \, dx dt$$

where

$$\mathbf{E} = \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.$$

In such a case the final expression for the extremal equation

$$\frac{\partial J_1}{\partial \mathbf{r}}$$

would stand for

$$\begin{aligned} & \frac{d(\rho \mathbf{v})}{dt} + \nabla P \\ & + \rho \mathbf{g} - K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} + K_1 \rho \mathbf{v} \times \mathbf{B} \\ & + \frac{1}{2} \frac{d\varepsilon_0(\rho(\mathbf{r}), \mathbf{r})}{d\mathbf{r}} \mathbf{E} \cdot \mathbf{E}. \\ = & \mathbf{0}, \end{aligned} \tag{14}$$

in $\Omega \times [0, T]$.

4 A note on the Energy equation

We define the power function P_w by

$$P_w = \left\{ -\frac{\partial J_2}{\partial \mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial t} : \text{subject to } \frac{\partial J_2}{\partial \mathbf{v}} = \frac{\partial J_2}{\partial \mathbf{A}} = \mathbf{0} \text{ and } \frac{\partial J_2}{\partial \rho} = \frac{\partial J_2}{\partial V} = 0 \right\}.$$

where

$$\begin{aligned} & J_2(\mathbf{r}, \mathbf{v}, \rho, \mathbf{A}, V) \\ = & \frac{1}{2} \int_0^T \int_{\Omega} \rho(\mathbf{r}(x, t), t) \mathbf{v}(\mathbf{r}(x, t), t) \cdot \mathbf{v}(\mathbf{r}(x, t), t) \, dxdt \\ & + \int_0^T \int_{\Omega} \lambda_1 \left(\mathbf{v}(\mathbf{r}(x, t), t) - \frac{\partial \mathbf{r}(x, t)}{\partial t} \right) \, dxdt + \int_0^T \int_{\Omega} \rho \mathbf{g} \cdot \mathbf{r} \, dxdt \\ & - K_1 \int_0^T \int_{\Omega} \rho \mathbf{A} \cdot \frac{\partial \mathbf{r}(x, t)}{\partial t} \, dxdt \\ & - K_1 \int_0^T \int_{\Omega} V(\mathbf{r}) \rho \, dxdt + \frac{1}{8\pi} \|\text{Curl } \mathbf{A} - \mathbf{B}_0\|_{0,2,\Omega \times [0,T]}^2 \\ & - \frac{1}{8\pi} \left\| \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right\|_{0,2,\Omega \times [0,T]}^2 \\ & - K_1 \int_0^T \int_{\Omega} \rho \mathbf{B}(x, t) \times \frac{\partial \mathbf{r}(x, t)}{\partial t} \cdot (\mathbf{r}(\varepsilon t, x) - \mathbf{r}_0(x)) \, dxdt. \end{aligned} \quad (15)$$

Such a Gâteaux differential

$$\frac{dJ_2}{d\mathbf{r}}$$

is related to the following variation of J_2 in $\mathbf{r}$, which lead us to the correct Euler-Lagrange equation,

$$\begin{aligned} & \left\langle \frac{dJ_2}{d\mathbf{r}}, \varphi \right\rangle_{L^2} = \hat{\delta}(J_2)_{\mathbf{r}}(\mathbf{r}, \mathbf{r}(\varepsilon t, x), \mathbf{v}, \rho, \mathbf{A}, V; \varphi) \\ = & \lim_{h \rightarrow 0} [J_2(\mathbf{r} + h\varphi, \mathbf{r}(\varepsilon t, x) + h\varphi, \mathbf{v}, \rho, \mathbf{A}, V) \\ & - J_2(\mathbf{r}, \mathbf{r}(\varepsilon t, x), \mathbf{v}, \rho, \mathbf{A}, V)] / h \\ = & \mathbf{0}, \quad \forall \varphi \in C_c^\infty(\Omega \times [0, T]; \mathbb{R}^3). \end{aligned} \quad (16)$$

The corresponding Euler-Lagrange equation stands for,

$$\frac{\partial \lambda_1}{\partial t} - \rho \mathbf{g} + K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} - K_1 \rho \mathbf{B} \times \mathbf{v} + \mathcal{O}(\varepsilon) = \mathbf{0},$$

in $\Omega \times [0, T]$.

With such a result and the restriction

$$\frac{\partial J_2}{\partial \mathbf{v}} = \mathbf{0},$$

which stands for

$$(\rho \mathbf{v}) + \lambda_1 = \mathbf{0},$$

so that from this and $P = \frac{d(\rho \hat{P})}{dt}$, we obtain

$$P_w = \left(\frac{d(\rho \mathbf{v})}{dt} + \rho \mathbf{g} - K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} + K_1 \rho \mathbf{B} \times \mathbf{v} \right) \cdot \mathbf{v} \quad (17)$$

in $\Omega \times [0, T]$.

Moreover, for an infinitesimal block volume $\tilde{V}$ of dimensions Δx_1 , Δx_2 and Δx_3 located at $(x_1, x_2, x_3, t) \in \Omega \times [0, T]$, we define for the pressure P , the following power variation,

$$\begin{aligned} \Delta \dot{W}_1 &= ((Pv_1)(x_1 + \Delta x_1, x_2, x_3, t) - (Pv_1)(x_1, x_2, x_3, t)) \Delta x_2 \Delta x_3 \\ &\quad + ((Pv_2)(x_1, x_2 + \Delta x_2, x_3, t) - (Pv_2)(x_1, x_2, x_3, t)) \Delta x_1 \Delta x_3 \\ &\quad + ((Pv_3)(x_1, x_2, x_3 + \Delta x_3, t) - (Pv_3)(x_1, x_2, x_3, t)) \Delta x_1 \Delta x_2 \\ &\approx \left(\frac{\partial(Pv_1)}{\partial x_1} + \frac{\partial(Pv_2)}{\partial x_2} + \frac{\partial(Pv_3)}{\partial x_3} \right) \Delta x_1 \Delta x_2 \Delta x_3 \\ &= \operatorname{div} (P\mathbf{v}) \Delta x_1 \Delta x_2 \Delta x_3. \end{aligned} \quad (18)$$

Also, concerning the temperature effects, denoting the temperature field by T_e , we generically assume the Fourier Law,

$$\dot{\mathbf{Q}} = KA \frac{\partial T_e}{\partial \mathbf{n}},$$

for a general flat area A with a surface normal outward field denoted by $\mathbf{n}$.

Here $\dot{\mathbf{Q}}$ denotes the heat flux through A in a direction normal to A .

Concerning our elementary small volume $\tilde{V}$, we define the following heat variation $\Delta \dot{\mathbf{Q}}$, where

$$\begin{aligned} \Delta \dot{\mathbf{Q}} &= K_e \left(\frac{\partial T_e}{\partial x_1}(x_1 + \Delta x_1, x_2, x_3, t) - \frac{\partial T_e}{\partial x_1}(x_1, x_2, x_3) \right) \Delta x_2 \Delta x_3 \\ &\quad + K_e \left(\frac{\partial T_e}{\partial x_2}(x_1, x_2 + \Delta x_2, x_3, t) - \frac{\partial T_e}{\partial x_2}(x_1, x_2, x_3) \right) \Delta x_1 \Delta x_3 \\ &\quad + K_e \left(\frac{\partial T_e}{\partial x_3}(x_1, x_2, x_3 + \Delta x_3, t) - \frac{\partial T_e}{\partial x_3}(x_1, x_2, x_3) \right) \Delta x_1 \Delta x_2 \\ &\approx K_e \left(\frac{\partial^2 T_e}{\partial x_1^2} + \frac{\partial^2 T_e}{\partial x_2^2} + \frac{\partial^2 T_e}{\partial x_3^2} \right) \Delta x_1 \Delta x_2 \Delta x_3 \\ &= K_e \nabla^2 T_e \Delta x_1 \Delta x_2 \Delta x_3. \end{aligned} \quad (19)$$

for some appropriate real constant $K_e > 0$.

Finally, for the Power variation concerning the internal variables, we define

$$E_{in} = \frac{1}{2} \rho \frac{d\mathbf{r}_e}{dt} \cdot \frac{d\mathbf{r}_e}{dt},$$

where generically

$$\mathbf{r}_e = \mathbf{r}_e(\mathbf{r}(x, t), t),$$

refers to a internal energy displacement field.

Moreover, we set the restriction,

$$C_v T_e = \frac{1}{2} \frac{d\mathbf{r}_e}{dt} \cdot \frac{d\mathbf{r}_e}{dt}, \text{ in } \Omega \times [0, T].$$

Observe that

$$-\frac{dE_{in}}{d\mathbf{r}_e} \cdot \frac{d\mathbf{r}_e}{dt} = \frac{d\left(\rho \frac{d\mathbf{r}_e}{dt}\right)}{dt} \cdot \frac{d\mathbf{r}_e}{dt}. \quad (20)$$

Now assuming

$$\left| \frac{d\rho}{dt} \frac{d\mathbf{r}_e}{dt} \right| \ll \left| \rho \frac{d^2(\mathbf{r}_e)}{dt^2} \right|,$$

we obtain

$$\begin{aligned} -\frac{dE_{in}}{d\mathbf{r}_e} \cdot \frac{d\mathbf{r}_e}{dt} &= \frac{d\left(\rho \frac{d\mathbf{r}_e}{dt}\right)}{dt} \cdot \frac{d\mathbf{r}_e}{dt} \\ &\approx \rho \frac{d^2(\mathbf{r}_e)}{dt^2} \cdot \frac{d\mathbf{r}_e}{dt} \\ &= \frac{\rho}{2} \frac{d\left(\frac{d\mathbf{r}_e}{dt} \cdot \frac{d\mathbf{r}_e}{dt}\right)}{dt} \\ &= \rho \frac{d(C_v T_e)}{dt}. \end{aligned} \quad (21)$$

Denoting

$$\Delta P_{in} = \rho \frac{d(C_v T_e)}{dt} \Delta x_1 \Delta x_2 \Delta x_3,$$

from an application of the First Law of Thermodynamics for such an elementary volume $\tilde{V}$, we have

$$P_w \Delta x_1 \Delta x_2 \Delta x_3 + \Delta P_{in} \approx -\Delta \dot{W} + \Delta \dot{Q}.$$

Therefore, we have obtained

$$\begin{aligned}
 & P_w + \rho \frac{d(C_v T_e)}{dt} \\
 & \approx -\operatorname{div}(P \mathbf{v}) + K_e \nabla^2 T_e \\
 & = -\nabla P \cdot \mathbf{v} - P \operatorname{div} \mathbf{v} + K_e \nabla^2 T_e.
 \end{aligned} \tag{22}$$

Now observe that

$$P_w + \nabla P \cdot \mathbf{v} = \left(\frac{d(\rho \mathbf{v})}{dt} + \nabla P + \rho \mathbf{g} - K_1 \frac{\partial(\rho \mathbf{A})}{\partial t} + K_1 \rho \mathbf{B} \times \mathbf{v} \right) \cdot \mathbf{v} = 0,$$

so that, we have got

$$\rho \frac{d(C_v T_e)}{dt} = K_e \nabla^2 T_e - P \operatorname{div} \mathbf{v}.$$

which is a standard energy equation found in the fluid mechanics books, for an inviscid compressible case.

Remark 4.1. *Generically, for an ideal gas, we will define the entropy function by*

$$S = -K_B \frac{\rho}{\rho_0} \ln \left(\frac{\rho}{e \rho_0} \right),$$

where ρ is the gas density field, ρ_0 is a reference density field, $K_B > 0$ is an appropriate real constant and e is the natural exponential basis.

Observe that for an ideal gas, we have

$$P = R \rho T_e,$$

where P is the pressure field, T_e is the temperature one and $R > 0$ is an appropriate constant.

From such results and definitions, we obtain,

$$\begin{aligned}
 S &= -K_B \frac{\rho}{\rho_0} \ln \left(\frac{\rho}{e \rho_0} \right) \\
 &= -K_B \frac{\rho}{\rho_0} \ln \left(\frac{P}{e R T_e \rho_0} \right) \\
 &= -K_B \frac{\rho}{\rho_0} \ln \left(\frac{P R T_0}{e R T_e P_0} \right) \\
 &= -K_B \frac{\rho}{\rho_0} \ln \left(\frac{P}{e P_0} \frac{T_0}{T_e} \right) \\
 &= -K_B \frac{\rho}{\rho_0} \ln \left(\frac{P}{e P_0} \right) + K_B \frac{\rho}{\rho_0} \ln \left(\frac{T_e}{T_0} \right)
 \end{aligned} \tag{23}$$

This final formula is similar to those found in the standard physics undergraduate books.

Including such an entropy function part

$$E_S = T_e S,$$

we obtain the final form of the energy equation,

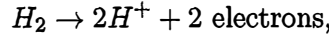
$$\rho \frac{d(C_v T_e)}{dt} = K_e \nabla^2 T_e - P \operatorname{div} \mathbf{v} + \frac{dE_S}{dt}.$$

The objective of this section is complete.

5 On the ionization chemical reaction and the corresponding plasma generation

In this section, we develop in details the concerning variational formulation for a compressible case.

We emphasize such a set $\Omega \subset \mathbb{R}^3$ stands for a control volume, in which along a time interval $[0, T]$, where $t \in [0, T]$ denotes time, a gaseous hydrogen molecular system will be ionized, through its exposition to a high scalar temperature field and an appropriate electric one, as it is indicated in the following reaction equation



generating a corresponding plasma.

We recall that a hydrogen (protium) molecule is comprised by two hydrogen (protium) atoms, with an electron and one proton each one (no neutrons).

5.1 The mathematical model

Denoting by m_e, m_p the masses of one electron and one proton, respectively, we start by defining the following hydrogen molecular density

$$\begin{aligned} \hat{\rho}_{H_2}(x, y, z, t) &= (|\phi_{e1}(x, y, z, t)|^2 + |\phi_{p1}(x, y, z, t)|^2) \\ &\quad \times \frac{|\phi_{A1}(y, z, t)|^2}{2m_e + 2m_p} \rho_{H_2}(z, t) \\ &\quad + (|\phi_{e2}(x, y, z, t)|^2 + |\phi_{p2}(x, y, z, t)|^2) \\ &\quad \times \frac{|\phi_{A2}(y, z, t)|^2}{2m_e + 2m_p} \rho_{H_2}(z, t). \end{aligned} \tag{24}$$

The corresponding ion densities $\hat{\rho}_{H_a^+}$ and $\hat{\rho}_{H_b^+}$ are defined by

$$\begin{aligned}\hat{\rho}_{H_a^+}(x, y, z, t) &= (|\phi_{p_3}(x, y, z, t)|^2) \\ &\times |\phi_{A_3}(y, z, t)|^2 \frac{\rho_{H_a^+}(z, t)}{m_p}\end{aligned}\quad (25)$$

and

$$\begin{aligned}\hat{\rho}_{H_b^+}(x, y, z, t) &= (|\phi_{p_4}(x, y, z, t)|^2) \\ &\times |\phi_{A_4}(y, z, t)|^2 \frac{\rho_{H_b^+}(z, t)}{m_p}\end{aligned}\quad (26)$$

In this text, we consider ρ_{H_2} , $\rho_{H_a^+}$, $\rho_{H_b^+}$, ϕ_{H_2} , $\phi_{H_a^+}$ and $\phi_{H_b^+}$ as independent variables which satisfy the following restrictions,

$$\rho_{H_2}(z, t) = |\phi_{H_2}(z, t)|^2.$$

$$\rho_{H_a^+}(z, t) = |\phi_{H_a^+}(z, t)|^2,$$

and

$$\rho_{H_b^+}(z, t) = |\phi_{H_b^+}(z, t)|^2,$$

respectively.

For the free electronic fields resulting from such an ionization, we denote

$$\rho_{e_3}(z, t) = |\phi_{e_3}(z, t)|^2,$$

and

$$\rho_{e_4}(z, t) = |\phi_{e_4}(z, t)|^2,$$

$$\rho_e(z, t) = \rho_{e_3}(z, t) + \rho_{e_4}(z, t).$$

Moreover, we denote

$$\hat{\rho}_{e_1}(x, y, z, t) = (|\rho_{e_1}(x, y, z, t)|^2) |\phi_{A_1}(y, z, t)|^2 \frac{|\phi_{H_2}(z, t)|^2}{2m_e + 2m_p},$$

$$\hat{\rho}_{e_2}(x, y, z, t) = (|\phi_{e_2}(x, y, z, t)|^2) |\phi_{A_2}(y, z, t)|^2 \frac{|\phi_{H_2}(z, t)|^2}{2m_e + 2m_p},$$

$$\hat{\rho}_{p_1}(x, y, z, t) = (|\phi_{p_1}(x, y, z, t)|^2) |\phi_{A_1}(y, z, t)|^2 \frac{|\phi_{H_2}(z, t)|^2}{2m_e + 2m_p},$$

and

$$\hat{\rho}_{p_2}(x, y, z, t) = (|\phi_{p_2}(x, y, z, t)|^2) |\phi_{A_2}(y, z, t)|^2 \frac{|\phi_{H_2}(z, t)|^2}{2m_e + 2m_p},$$

$$\hat{\rho}_{p_3}(x, y, z, t) = (|\phi_{p_3}(x, y, z, t)|^2) |\phi_{A_3}(y, z, t)|^2 \frac{|\phi_{H_a^+}(z, t)|^2}{m_p},$$

$$\hat{\rho}_{p_4}(x, y, z, t) = (|\phi_{p_4}(x, y, z, t)|^2) |\phi_{A_4}(y, z, t)|^2 \frac{|\phi_{H_b^+}(z, t)|^2}{m_p},$$

$$\rho_{e_1}(z, t) = \int_{\Omega} \int_{\Omega} \hat{\rho}_{e_1}(x, y, z, t) \, dx \, dy,$$

$$\rho_{e_2}(z, t) = \int_{\Omega} \int_{\Omega} \hat{\rho}_{e_2}(x, y, z, t) \, dx \, dy,$$

$$\rho_{p_1}(z, t) = \int_{\Omega} \int_{\Omega} \hat{\rho}_{p_1}(x, y, z, t) \, dx \, dy,$$

$$\rho_{p_2}(z, t) = \int_{\Omega} \int_{\Omega} \hat{\rho}_{e_2}(x, y, z, t) \, dx \, dy,$$

$$\rho_{p_3}(z, t) = \int_{\Omega} \int_{\Omega} \hat{\rho}_{p_3}(x, y, z, t) \, dx \, dy,$$

$$\rho_{p_4}(z, t) = \int_{\Omega} \int_{\Omega} \hat{\rho}_{p_4}(x, y, z, t) \, dx \, dy.$$

Here the subscripts e_j , p_j , A_j refer to a density of electrons, protons and atoms for $j \in \{1, 2, 3, 4\}$.

We define also the following function spaces,

$$\begin{aligned} V_1 = & \{(\phi_{e_1}, \phi_{e_2}) \in W^{1,2}(\Omega \times \Omega \times \Omega \times [0, T]; \mathbb{C}^2) : \\ & \nabla_x \phi_{e_j} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \times \Omega \times \Omega \times [0, T], \\ & \nabla_y \phi_{e_j} \cdot \mathbf{n} = 0, \text{ on } \Omega \times \partial\Omega \times \Omega \times [0, T], \\ & \nabla_z \phi_{e_j} \cdot \mathbf{n} = 0, \text{ on } \Omega \times \Omega \times \partial\Omega \times [0, T], \\ & \frac{\partial \phi_{e_j}(x, y, z, 0)}{\partial t} = 0 \\ & \text{and } \frac{\partial \phi_{e_j}(x, y, z, T)}{\partial t} = 0, \text{ in } \Omega \times \Omega \times \Omega, \forall j \in \{1, 2\} \}, \end{aligned} \quad (27)$$

$$\begin{aligned}
 V_2 = & \{(\phi_{p_1}, \phi_{p_2}) \in W^{1,2}(\Omega \times \Omega \times \Omega \times [0, T]; \mathbb{C}^2) : \\
 & \nabla_x \phi_{p_j} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \times \Omega \times \Omega \times [0, T], \\
 & \nabla_y \phi_{p_j} \cdot \mathbf{n} = 0, \text{ on } \Omega \times \partial\Omega \times \Omega \times [0, T], \\
 & \nabla_z \phi_{p_j} \cdot \mathbf{n} = 0, \text{ on } \Omega \times \Omega \times \partial\Omega \times [0, T], \\
 & \frac{\partial \phi_{p_j}(x, y, z, 0)}{\partial t} = 0 \\
 & \text{and } \frac{\partial \phi_{p_j}(x, y, z, T)}{\partial t} = 0, \text{ in } \Omega \times \Omega \times \Omega, \forall j \in \{1, 2\} \}, \quad (28)
 \end{aligned}$$

$$\begin{aligned}
 V_3 = & \{(\phi_{e_3}, \phi_{e_4}) \in W^{1,2}(\Omega \times [0, T]; \mathbb{C}^2) : \nabla_z \phi_{e_j} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \times [0, T], \\
 & \frac{\partial \phi_{e_j}(z, 0)}{\partial t} = 0 \text{ and } \frac{\partial \phi_{e_j}(z, T)}{\partial t} = 0, \\
 & \text{in } \Omega \times \Omega \times \Omega, \forall j \in \{3, 4\} \}, \quad (29)
 \end{aligned}$$

$$\begin{aligned}
 V_4 = & \{(\phi_{p_3}, \phi_{p_4}) \in W^{1,2}(\Omega \times \Omega \times \Omega \times [0, T]; \mathbb{C}^2) : \\
 & \nabla_x \phi_{p_j} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \times \Omega \times \Omega \times [0, T], \\
 & \nabla_y \phi_{p_j} \cdot \mathbf{n} = 0, \text{ on } \Omega \times \partial\Omega \times \Omega \times [0, T], \\
 & \nabla_z \phi_{p_j} \cdot \mathbf{n} = 0, \text{ on } \Omega \times \Omega \times \partial\Omega \times [0, T], \\
 & \frac{\partial \phi_{p_j}(x, y, z, 0)}{\partial t} = 0 \\
 & \text{and } \frac{\partial \phi_{p_j}(x, y, z, T)}{\partial t} = 0, \text{ in } \Omega \times \Omega \times \Omega, \forall j \in \{3, 4\} \}, \quad (30)
 \end{aligned}$$

$$\begin{aligned}
 V_5 = & \{(\phi_{A_1}, \phi_{A_2}) \in W^{1,2}(\Omega \times \Omega \times [0, T]; \mathbb{C}^2) : \\
 & \nabla_y \phi_{A_j} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \times \Omega \times [0, T], \\
 & \nabla_z \phi_{A_j} \cdot \mathbf{n} = 0, \text{ on } \Omega \times \partial\Omega \times [0, T], \\
 & \frac{\partial \phi_{A_j}(y, z, 0)}{\partial t} = 0 \\
 & \text{and } \frac{\partial \phi_{A_j}(y, z, T)}{\partial t} = 0, \text{ in } \Omega \times \Omega, \forall j \in \{1, 2\} \}, \quad (31)
 \end{aligned}$$

$$\begin{aligned}
V_6 = & \{(\phi_{A_3}, \phi_{A_4}) \in W^{1,2}(\Omega \times \Omega \times [0, T]; \mathbb{C}^2) : \\
& \nabla_y \phi_{A_j} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \times \Omega \times [0, T], \\
& \nabla_z \phi_{A_j} \cdot \mathbf{n} = 0, \text{ on } \Omega \times \partial\Omega \times [0, T], \\
& \frac{\partial \phi_{A_j}(y, z, 0)}{\partial t} = 0 \\
& \text{and } \frac{\partial \phi_{A_j}(y, z, T)}{\partial t} = 0, \text{ in } \Omega \times \Omega, \forall j \in \{3, 4\} \}, \quad (32)
\end{aligned}$$

$$\begin{aligned}
V_7 = & \{(\phi_{H_2}, \phi_{H_a^+}, \phi_{H_b^+}) \in W^{1,2}(\Omega \times [0, T]; \mathbb{C}^3) : \\
& (\phi_{H_2}, \phi_{H_a^+}, \phi_{H_b^+}) = \mathbf{0} \text{ on } \partial(\Omega_1 \cup \Omega_2) \times [0, T], \\
& (\phi_{H_2}(z, t), \phi_{H_a^+}(z, t), \phi_{H_b^+}(z, t)) = (\phi_{(H_2)_{in}}(z, t), 0, 0) \text{ on } \partial\Omega_{in} \times [0, T] \\
& \text{and } (\phi_{H_2}(z, 0), \phi_{H_a^+}(z, 0), \phi_{H_b^+}(z, 0)) = (\phi_0, 0, 0) \text{ in } \Omega \}, \quad (33)
\end{aligned}$$

where

$$\partial\Omega = \partial\Omega_{in} \cup \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_{out}.$$

Moreover we suppose such a union is quasi-disjoint and through $\partial\Omega_{in}$ and $\partial\Omega_{out}$ are allowed the entering and leaving of gaseous mass, respectively.

5.2 The position fields

We start by denoting the macroscopic molecular hydrogen position field as

$$\mathbf{r}_{H_2} : \Omega \times [0, T] \rightarrow \mathbb{R}^3.$$

The atomic position fields are defined by

$$\mathbf{r}_{A_j}(x, t) = \mathbf{r}_{H_2}(x, t) + \hat{\mathbf{r}}_{A_j}(\mathbf{r}_{H_2}(x, t), t),$$

for $j \in \{1, 2\}$, and

$$\mathbf{r}_{A_3}(x, t) = \mathbf{r}_{H_a^+}(x, t) + \hat{\mathbf{r}}_{A_3}(\mathbf{r}_{H_a^+}(x, t), t),$$

$$\mathbf{r}_{A_4}(x, t) = \mathbf{r}_{H_b^+}(x, t) + \hat{\mathbf{r}}_{A_4}(\mathbf{r}_{H_b^+}(x, t), t),$$

where

$$\mathbf{r}_{H_a^+} : \Omega \times [0, T] \rightarrow \mathbb{R}^4,$$

and

$$\mathbf{r}_{H_b^+} : \Omega \times [0, T] \rightarrow \mathbb{R}^4$$

denote the macroscopic ion position field for the ions H_a^+ and H_b^+ , respectively.

Moreover, in a relativistic context, for the electronic fields

$$\hat{\mathbf{r}}_{e_j}(x, t) = \mathbf{r}_{A_j}(x, t) + \tilde{\mathbf{r}}_{e_j}(\mathbf{r}_{A_j}(x, t), t) = (\mathbf{r}_{H_2}(x, t), t) + (ct, \mathbf{r}_{e_j}(\mathbf{r}_{H_2}(x, t))),$$

for $j \in \{1, 2\}$.

For the proton fields,

$$\hat{\mathbf{r}}_{p_j}(x, t) = \mathbf{r}_{A_j}(x, t) + \tilde{\mathbf{r}}_{p_j}(\mathbf{r}_{A_j}(x, t), t) = (\mathbf{r}_{H_2}(x, t), t) + (ct, \mathbf{r}_{p_j}(\mathbf{r}_{H_2}(x, t))),$$

for $j \in \{1, 2\}$.

Moreover for the ion proton fields

$$\hat{\mathbf{r}}_{p_3}(x, t) = \mathbf{r}_{A_3}(x, t) + \tilde{\mathbf{r}}_{p_3}(\mathbf{r}_{A_3}(x, t), t) = (\mathbf{r}_{H_a^+}(x, t), t) + (ct, \mathbf{r}_{p_3}(\mathbf{r}_{H_a^+}(x, t))),$$

and

$$\hat{\mathbf{r}}_{p_4}(x, t) = \mathbf{r}_{A_4}(x, t) + \tilde{\mathbf{r}}_{p_4}(\mathbf{r}_{A_4}(x, t), t) = (\mathbf{r}_{H_b^+}(x, t), t) + (ct, \mathbf{r}_{p_4}(\mathbf{r}_{H_b^+}(x, t))),$$

Here $c > 0$ denotes the speed of light.

Finally, generically we shall denote,

$$\mathbf{v}_{H_2} = \mathbf{v}_{H_2}(\mathbf{r}_{H_2}(x, t), t),$$

$$\mathbf{v}_{H_a^+} = \mathbf{v}_{H_a^+}(\mathbf{r}_{H_a^+}(x, t), t),$$

$$\mathbf{v}_{H_b^+} = \mathbf{v}_{H_b^+}(\mathbf{r}_{H_b^+}(x, t), t),$$

$$\mathbf{v}_{e_3} = \mathbf{v}_{e_3}(\mathbf{r}_{e_3}(x, t), t)$$

and

$$\mathbf{v}_{e_4} = \mathbf{v}_{e_4}(\mathbf{r}_{e_4}(x, t), t),$$

with a similar corresponding notation for the remaining particles and atoms.

Here we define the following function spaces,

$$\begin{aligned} V_8 = & \left\{ \mathbf{r} = (\mathbf{r}_{H_2}, \mathbf{r}_{H_a^+}, \mathbf{r}_{H_b^+}) \in W^{1,2}(\Omega \times [0, T]; \mathbb{R}^{3 \times 3}) : \right. \\ & \mathbf{r} = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, T] \\ & \left. \text{and } \mathbf{r}(z, 0) = \mathbf{r}_0 \text{ in } \Omega \right\}, \end{aligned} \tag{34}$$

and

$$(\mathbf{r}_{e_1}, \mathbf{r}_{e_2}) \in V_9 = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{4 \times 2}),$$

$$\begin{aligned}
(\mathbf{r}_{p_1}, \mathbf{r}_{p_2}) &\in V_{10} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{4 \times 2}), \\
(\mathbf{v}_{p_3}, \mathbf{v}_{p_4}) &\in V_{11} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{4 \times 2}), \\
(\mathbf{r}_{e_3}, \mathbf{r}_{e_4}) &\in V_{12} = W^{1,2}(\Omega \times [0, T]; \mathbb{R}^{4 \times 2}), \\
(\mathbf{r}_{A_1}, \mathbf{r}_{A_2}) &\in V_{13} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{3 \times 2}), \\
(\mathbf{r}_{A_3}, \mathbf{r}_{A_3}) &\in V_{14} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{3 \times 2}).
\end{aligned}$$

$$\begin{aligned}
V_{15} = \{ &\mathbf{v} = (\mathbf{v}_{H_2}, \mathbf{v}_{H_a^+}, \mathbf{v}_{H_b^+}) \in W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{3 \times 3}) : \mathbf{v} = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, T] \\
&\text{and } \mathbf{v}(\mathbf{r}, 0) = \mathbf{v}_0 \text{ in } \Omega\}, \tag{35}
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{v}_{e_1}, \mathbf{v}_{e_2}) &\in V_{16} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{4 \times 2}), \\
(\mathbf{v}_{p_1}, \mathbf{v}_{p_2}) &\in V_{17} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{4 \times 2}), \\
(\mathbf{v}_{p_3}, \mathbf{v}_{p_4}) &\in V_{18} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{4 \times 2}), \\
(\mathbf{v}_{e_3}, \mathbf{v}_{e_4}) &\in V_{19} = W^{1,2}(\Omega \times [0, T]; \mathbb{R}^{4 \times 2}), \\
(\mathbf{v}_{A_1}, \mathbf{v}_{A_2}) &\in V_{20} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{3 \times 2}), \\
(\mathbf{v}_{A_3}, \mathbf{v}_{A_3}) &\in V_{21} = W^{1,2}(\mathbb{R}^3 \times [0, T]; \mathbb{R}^{3 \times 2}).
\end{aligned}$$

5.3 About the system constraints

Denoting by $m_{H_2}(0)$ the total system mass at $t = 0$, we have the following constraint

$$\begin{aligned}
m_{H_2}(0) &= \int_{\Omega} \rho_{H_2}(z, 0) dz \\
&= \int_{\Omega} \rho_{H_2}(z, t) dz + \int_0^t \int_{\partial\Omega} \rho_{H_2}(z, \tau) \mathbf{v}_{H_2} \cdot \mathbf{n} dS d\tau \\
&\quad + \int_{\Omega} \rho_{H_a^+}(z, t) dz + \int_0^t \int_{\partial\Omega} \rho_{H_a^+}(z, \tau) \mathbf{v}_{H_a^+} \cdot \mathbf{n} dS d\tau \\
&\quad + \int_{\Omega} \rho_{H_b^+}(z, t) dz + \int_0^t \int_{\partial\Omega} \rho_{H_b^+}(z, \tau) \mathbf{v}_{H_b^+} \cdot \mathbf{n} dS d\tau \\
&\quad + \int_{\Omega} \rho_{e_3}(z, t) dz + \int_0^t \int_{\partial\Omega} \rho_{e_3}(z, \tau) \mathbf{v}_{e_3} \cdot \mathbf{n} dS d\tau \\
&\quad + \int_{\Omega} \rho_{e_4}(z, t) dz + \int_0^t \int_{\partial\Omega} \rho_{e_4}(z, \tau) \mathbf{v}_{e_4} \cdot \mathbf{n} dS d\tau \tag{36}
\end{aligned}$$

We have also the following constraint

$$\begin{aligned} & \int_{\Omega} \rho_{e_3}(z, t) dz + \int_0^t \int_{\partial\Omega} \rho_{e_3}(z, \tau) \mathbf{v}_{e_3} \cdot \mathbf{n} dS d\tau \\ &= \int_{\Omega} \rho_{e_4}(z, t) dz + \int_0^t \int_{\partial\Omega} \rho_{e_4}(z, \tau) \mathbf{v}_{e_4} \cdot \mathbf{n} dS d\tau \end{aligned} \quad (37)$$

Here $\mathbf{n}$ denotes the outward normal field to $\partial\Omega = S$.

Relating the kinetics energy, we define the following functionals

$$\begin{aligned} E_{C_1} &= -\frac{1}{2} \int_0^T \int_{\Omega} \frac{\rho_{e_1}(z, t)}{\sqrt{1 - \frac{v_{e_1}^2}{c^2}}} \frac{d\tilde{\mathbf{r}}_{e_1}}{dt} \cdot \frac{d\tilde{\mathbf{r}}_{e_1}}{dt} \sqrt{-g^{e_1}} |\det X'_{H_2}| dz dt \\ &= \frac{1}{2} \int_0^T \int_{\Omega} \frac{\rho_{e_1}(z, t)}{\sqrt{1 - \frac{v_{e_1}^2}{c^2}}} (c^2 - v_{e_1}^2) \sqrt{-g^{e_1}} |\det X'_{H_2}| dz dt \\ &= \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{e_1}(z, t) \sqrt{c^2 - v_{e_1}^2} \sqrt{-g^{e_1}} |\det X'_{H_2}| dz dt \\ &= \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{e_1}(z, t) \sqrt{c^2 - g_{jk}^{e_1} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{e_1}} |\det X'_{H_2}| dz dt, \end{aligned}$$

where generically, we have denoted, $X_0 = t$,

$$g_{jk}^{e_1} = \frac{\partial \mathbf{r}_{e_1}}{\partial (X_{H_2})_j} \frac{\partial \mathbf{r}_{e_1}}{\partial (X_{H_2})_k}$$

$g^{e_1} = \det\{g_{jk}^{e_1}\}$, where

$$a \cdot b = -a_0 b_0 + \sum_{k=1}^3 a_k b_k$$

for

$$a = (a_0, a_1, a_2, a_3)$$

and

$$b = (b_0, b_1, b_2, b_3) \in \mathbb{R}^4.$$

Similarly, we may define

$$E_{C_2} = \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{p_1}(z, t) \sqrt{c^2 - g_{jk}^{p_1} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{p_1}} |\det X'_{H_2}| dz dt \quad (38)$$

and also,

$$E_{C_3} = \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{e_2}(z, t) \sqrt{c^2 - g_{jk}^{e_2} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{e_2}} |\det X'_{H_2}| dz dt \quad (39)$$

Furthermore, we may define

$$E_{C_4} = \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{p_2}(z, t) \sqrt{c^2 - g_{jk}^{p_2} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{p_2}} |\det X'_{H_2}| dz dt \quad (40)$$

Similarly, for the ions particles

$$E_{C_5} = \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{p_3}(z, t) \sqrt{c^2 - g_{jk}^{p_3} \frac{\partial(X_{H_a^+})_j}{\partial t} \frac{\partial(X_{H_a^+})_k}{\partial t}} \sqrt{-g^{p_3}} |\det X'_{H_a^+}| dz dt \quad (41)$$

$$E_{C_6} = \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{p_4}(z, t) \sqrt{c^2 - g_{jk}^{p_4} \frac{\partial(X_{H_b^+})_j}{\partial t} \frac{\partial(X_{H_b^+})_k}{\partial t}} \sqrt{-g^{p_4}} |\det X'_{H_b^+}| dz dt \quad (42)$$

$$E_{C_7} = \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{e_3}(z, t) \sqrt{c^2 - v_{e_3}^2} \sqrt{-g^{e_3}} dz dt, \quad (43)$$

$$E_{C_8} = \frac{1}{2} \int_0^T \int_{\Omega} c \rho_{e_4}(z, t) \sqrt{c^2 - v_{e_4}^2} \sqrt{-g^{e_4}} dz dt. \quad (44)$$

For the interacting and self-interacting energies, in a non-linear Schrödinger equation context, we define the following functionals,

$$\begin{aligned}
E_{in} = & \frac{\alpha_1}{2} \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} |\hat{\rho}_{H_2}(x, y, z, t)|^2 dx dy dz dt \\
& + \frac{\alpha_2}{2} \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} |\hat{\rho}_{H_a^+}(x, y, z, t)|^2 dx dy dz dt \\
& + \frac{\alpha_3}{2} \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} |\hat{\rho}_{H_b^+}(x, y, z, t)|^2 dx dy dz dt \\
& + \frac{\alpha_4}{2} \int_0^T \int_{\Omega} |\rho_{e_3}(z, t)|^2 dz dt \\
& + \frac{\alpha_5}{2} \int_0^T \int_{\Omega} |\rho_{e_4}(z, t)|^2 dz dt \\
& - \frac{\beta_1}{2} \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{H_2}(x, y, z, t) dx dy dz dt \\
& - \frac{\beta_2}{2} \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{H_a^+}(x, y, z, t) dx dy dz dt \\
& - \frac{\beta_3}{2} \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{H_b^+}(x, y, z, t) dx dy dz dt \\
& - \frac{\beta_4}{2} \int_0^T \int_{\Omega} \rho_{e_3}(z, t) dz dt \\
& - \frac{\beta_5}{2} \int_0^T \int_{\Omega} \rho_{e_4}(z, t) dz dt,
\end{aligned} \tag{45}$$

On the other hand, for the terms involving the magnetic potential, we define

$$\begin{aligned}
E_{M_1} = & \frac{1}{2} \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \left(\left(\frac{\gamma_1}{2m_e + 2m_p} |\nabla \phi_{e_1}(x, y, z, t) - i\rho_1 \mathbf{A}(z, t) \phi_{e_1}(x, y, z, t)|^2 \right. \right. \\
& + \frac{\gamma_2}{2m_e + 2m_p} |\nabla \phi_{p_1}(x, y, z, t) - i\rho_2 \mathbf{A}(z, t) \phi_{p_1}(x, y, z, t)|^2 \Big) \\
& \times |\phi_{A_1}(y, z, t)|^2 |\phi_{H_2}(z, t)|^2 \\
& + \left(\frac{\gamma_4}{2m_e + 2m_p} |\nabla \phi_{e_2}(x, y, z, t) - i\rho_4 \mathbf{A}(z, t) \phi_{e_2}(x, y, z, t)|^2 \right. \\
& + \frac{\gamma_5}{2m_e + 2m_p} |\nabla \phi_{p_2}(x, y, z, t) - i\rho_5 \mathbf{A}(z, t) \phi_{p_2}(x, y, z, t)|^2 \Big) \\
& \times |\phi_{A_2}(y, z, t)|^2 |\phi_{H_2}(z, t)|^2 dx dy dz dt.
\end{aligned} \tag{46}$$

Here, generically we have denoted

$$\nabla\phi(x, y, z, t) = (i\phi_t, \phi_{x_1}, \phi_{x_2}, \phi_{x_3}, \phi_{y_1}, \phi_{y_2}, \phi_{y_3}, \phi_{z_1}, \phi_{z_2}, \phi_{z_3})$$

where $i \in \mathbb{C}$ denotes the imaginary unit, so that

$$\nabla\phi \cdot \nabla\phi = -\phi_t^2 + \sum_{j=1}^3 (\phi_{x_j}^2 + \phi_{y_j}^2 + \phi_{z_j}^2).$$

$$\begin{aligned} E_{M_2} = & \frac{1}{2} \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} ((\gamma_7 |\nabla\phi_{p_3}(x, y, z, t) - i\rho_7 \mathbf{A}(z, t)\phi_{p_3}(x, y, z, t)|^2) \\ & \times |\phi_{A_3}(y, z, t)|^2 \frac{|\phi_{H_a^+}^2(z, t)|^2}{m_p}) dx dy dz dt \end{aligned} \quad (47)$$

and

$$\begin{aligned} E_{M_3} = & \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} ((\gamma_9 |\nabla\phi_{p_4}(x, y, z, t) - i\rho_9 \mathbf{A}(z, t)\phi_{p_4}(x, y, z, t)|^2) \\ & \times |\phi_{A_4}(y, z, t)|^2 \frac{|\phi_{H_b^+}^2(z, t)|^2}{m_p}) dx dy dz dt, \end{aligned} \quad (48)$$

$$E_{M_4} = \frac{\gamma_{11}}{2} \int_0^T \int_{\Omega} |\nabla\phi_{e_3}(z, t) - i\rho_{11} \mathbf{A}(z, t)\phi_{e_3}(z, t)|^2 dz dt,$$

and

$$E_{M_5} = \frac{\gamma_{12}}{2} \int_0^T \int_{\Omega} |\nabla\phi_{e_4}(z, t) - i\rho_{12} \mathbf{A}(z, t)\phi_{e_4}(z, t)|^2 dz dt.$$

Concerning the restrictions, considering the link between the two electrons, we define also the following functionals,

$$\begin{aligned} J_{Awx_1} = & \int_0^T \int_{\Omega} \int_{\Omega} E_1(y, z, t) \\ & \times \left(\int_{\Omega} (|\phi_{e_1}(x, y, z, t)|^2 + |\phi_{e_2}(x, y, z, t)|^2) dx - 2m_e \right) dy dz dt, \end{aligned} \quad (49)$$

$$J_{Awx_2} = \int_0^T \int_{\Omega} \int_{\Omega} E_2(y, z, t) \left(\int_{\Omega} |\phi_{p_1}(x, y, z, t)|^2 dx - m_p \right) dy dz dt, \quad (50)$$

$$J_{Aux_4} = \int_0^T \int_{\Omega} \int_{\Omega} E_5(y, z, t) \left(\int_{\Omega} |\phi_{p_2}(x, y, z, t)|^2 dx - m_p \right) dy dz dt, \quad (51)$$

$$J_{Aux_5} = \int_0^T \int_{\Omega} \int_{\Omega} E_7(y, z, t) \left(\int_{\Omega} |\phi_{p_3}(x, y, z, t)|^2 dx - m_p \right) dy dz dt, \quad (52)$$

$$J_{Aux_6} = \int_0^T \int_{\Omega} \int_{\Omega} E_9(y, z, t) \left(\int_{\Omega} |\phi_{p_4}(x, y, z, t)|^2 dx - m_p \right) dy dz dt, \quad (53)$$

$$J_{Aux_7} = \int_0^T \int_{\Omega} \int_{\Omega} E_{11}(z, t) \left(\int_{\Omega} |\phi_{A_1}(y, z, t)|^2 dy - 1 \right) dz dt, \quad (54)$$

$$J_{Aux_8} = \int_0^T \int_{\Omega} \int_{\Omega} E_{12}(z, t) \left(\int_{\Omega} |\phi_{A_2}(y, z, t)|^2 dy - 1 \right) dz dt, \quad (55)$$

$$J_{Aux_9} = \int_0^T \int_{\Omega} \int_{\Omega} E_{13}(z, t) \left(\int_{\Omega} |\phi_{A_3}(y, z, t)|^2 dy - 1 \right) dz dt, \quad (56)$$

$$J_{Aux_{10}} = \int_0^T \int_{\Omega} \int_{\Omega} E_{14}(z, t) \left(\int_{\Omega} |\phi_{A_4}(y, z, t)|^2 dy - 1 \right) dz dt, \quad (57)$$

$$\begin{aligned} J_{Aux_{11}} = & \int_0^T E_{15}(t) \left(\int_{\Omega} \rho_{H_2}(z, 0) dz \right. \\ & - \int_{\Omega} \rho_{H_2}(z, t) dz - \int_0^t \int_{\partial\Omega} \rho_{H_2}(z, \tau) \mathbf{v}_{H_2} \cdot \mathbf{n} dS d\tau \\ & - \int_{\Omega} \rho_{H_a^+}(z, t) dz - \int_0^t \int_{\partial\Omega} \rho_{H_a^+}(z, \tau) \mathbf{v}_{H_a^+} \cdot \mathbf{n} dS d\tau \\ & - \int_{\Omega} \rho_{H_b^+}(z, t) dz - \int_0^t \int_{\partial\Omega} \rho_{H_b^+}(z, \tau) \mathbf{v}_{H_b^+} \cdot \mathbf{n} dS d\tau \\ & - \int_{\Omega} \rho_{e_3}(z, t) dz - \int_0^t \int_{\partial\Omega} \rho_{e_3}(z, \tau) \mathbf{v}_{e_3} \cdot \mathbf{n} dS d\tau \\ & \left. - \int_{\Omega} \rho_{e_4}(z, t) dz - \int_0^t \int_{\partial\Omega} \rho_{e_4}(z, \tau) \mathbf{v}_{e_4} \cdot \mathbf{n} dS d\tau \right) dt \quad (58) \end{aligned}$$

We have also the following constraints

$$J_{Aux_{12}} = \int_0^T E_{16}(t) \left(\int_{\Omega} \rho_{e_3}(z, t) dz + \int_0^t \int_{\partial\Omega} \rho_{e_3}(z, \tau) \mathbf{v}_{e_3} \cdot \mathbf{n} dS d\tau \right. \\ \left. - \int_{\Omega} \rho_{e_4}(z, t) dz - \int_0^t \int_{\partial\Omega} \rho_{e_4}(z, \tau) \mathbf{v}_{e_4} \cdot \mathbf{n} dS d\tau \right) dt \quad (59)$$

$$J_{Aux_{13}} = \int_0^T \int_{\Omega} E_{17}(x, t) \left(\mathbf{v}_{H_2} - \frac{\partial \mathbf{r}_{H_2}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{14}} = \int_0^T \int_{\Omega} E_{18}(x, t) \left(\mathbf{v}_{H_a^+} - \frac{\partial \mathbf{r}_{H_a^+}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{15}} = \int_0^T \int_{\Omega} E_{19}(x, t) \left(\mathbf{v}_{H_b^+} - \frac{\partial \mathbf{r}_{H_b^+}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{16}} = \int_0^T \int_{\Omega} E_{20}(x, t) \left(\mathbf{v}_{e_1} - \frac{\partial \mathbf{r}_{e_1}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{17}} = \int_0^T \int_{\Omega} E_{21}(x, t) \left(\mathbf{v}_{p_1} - \frac{\partial \mathbf{r}_{p_1}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{18}} = \int_0^T \int_{\Omega} E_{23}(x, t) \left(\mathbf{v}_{e_2} - \frac{\partial \mathbf{r}_{e_2}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{19}} = \int_0^T \int_{\Omega} E_{24}(x, t) \left(\mathbf{v}_{p_1} - \frac{\partial \mathbf{r}_{p_2}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{20}} = \int_0^T \int_{\Omega} E_{26}(x, t) \left(\mathbf{v}_{p_3} - \frac{\partial \mathbf{r}_{p_3}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{22}} = \int_0^T \int_{\Omega} E_{28}(x, t) \left(\mathbf{v}_{p_4} - \frac{\partial \mathbf{r}_{p_4}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{23}} = \int_0^T \int_{\Omega} E_{30}(x, t) \left(\mathbf{v}_{e_3} - \frac{\partial \mathbf{r}_{e_3}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{24}} = \int_0^T \int_{\Omega} E_{31}(x, t) \left(\mathbf{v}_{e_4} - \frac{\partial \mathbf{r}_{e_4}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{25}} = \int_0^T \int_{\Omega} E_{32}(x, t) \left(\mathbf{v}_{A_1} - \frac{\partial \mathbf{r}_{A_1}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{26}} = \int_0^T \int_{\Omega} E_{33}(x, t) \left(\mathbf{v}_{A_2} - \frac{\partial \mathbf{r}_{A_2}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{27}} = \int_0^T \int_{\Omega} E_{34}(x, t) \left(\mathbf{v}_{A_3} - \frac{\partial \mathbf{r}_{A_3}(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_{28}} = \int_0^T \int_{\Omega} E_{35}(x, t) \left(\mathbf{v}_{A_4} - \frac{\partial \mathbf{r}_{A_4}(x, t)}{\partial t} \right) dx dt,$$

Defining $\rho(z, t) = \rho_{H_2}(z, t) + \rho_{H_a^+}(z, t) + \rho_{H_b^+}(z, t) + \rho_{e_3}(z, t) + \rho_{e_4}(z, t)$, we define also

$$\begin{aligned} J_{Aux_{29}} = & - \int_0^T \int_{\Omega} \hat{P} \left(\frac{d\rho}{dt} + \rho_{H_2} \operatorname{div} \mathbf{v}_{H_2} + \rho_{H_a^+} \operatorname{div} \mathbf{v}_{H_a^+} \right. \\ & \left. + \rho_{H_b^+} \operatorname{div} \mathbf{v}_{H_b^+} + \rho_{e_3} \operatorname{div} \mathbf{v}_{e_3} + \rho_{e_4} \operatorname{div} \mathbf{v}_{e_4} \right) dz dt \end{aligned} \quad (60)$$

Moreover, considering the pressure and temperature scalar fields, we define

$$P = \frac{d(\hat{P}\rho)}{dt},$$

$$P_{H_2} = \frac{d(\hat{P}\rho_{H_2})}{dt},$$

$$P_{H_a^+} = \frac{d(\hat{P}\rho_{H_a^+})}{dt},$$

$$P_{H_b^+} = \frac{d(\hat{P}\rho_{H_b^+})}{dt},$$

$$P_{e_3} = \frac{d(\hat{P}\rho_{e_3})}{dt},$$

$$P_{e_4} = \frac{d(\hat{P}\rho_{e_4})}{dt},$$

$$\begin{aligned}
& (C_v)_{H_2} \rho_{H_2} T_{H_2} \\
= & \frac{1}{2} c \rho_{e_1}(z, t) \sqrt{c^2 - g_{jk}^{e_1} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{e_1}} |\det X'_{H_2}| \\
& + \frac{1}{2} c \rho_{p_1}(z, t) \sqrt{c^2 - g_{jk}^{p_1} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{p_1}} |\det X'_{H_2}| \\
& + \frac{1}{2} c \rho_{e_2}(z, t) \sqrt{c^2 - g_{jk}^{e_2} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{e_2}} |\det X'_{H_2}| \\
& + \frac{1}{2} c \rho_{p_2}(z, t) \sqrt{c^2 - g_{jk}^{p_2} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{p_2}} |\det X'_{H_2}| \quad (61)
\end{aligned}$$

$$\begin{aligned}
& (C_v)_{H_a^+} \rho_{H_a^+} T_{H_a^+} \\
= & \frac{1}{2} c \rho_{p_3}(z, t) \sqrt{c^2 - g_{jk}^{p_3} \frac{\partial(X_{H_a^+})_j}{\partial t} \frac{\partial(X_{H_a^+})_k}{\partial t}} \sqrt{-g^{p_3}} |\det X'_{H_a^+}| \quad (62)
\end{aligned}$$

$$\begin{aligned}
& (C_v)_{H_b^+} \rho_{H_b^+} T_{H_b^+} \\
= & \frac{1}{2} c \rho_{p_4}(z, t) \sqrt{c^2 - g_{jk}^{p_4} \frac{\partial(X_{H_b^+})_j}{\partial t} \frac{\partial(X_{H_b^+})_k}{\partial t}} \sqrt{-g^{p_4}} |\det X'_{H_b^+}| \quad (63)
\end{aligned}$$

and

$$\begin{aligned}
(C_v)_e \rho_e T_e &= \frac{1}{2} c \rho_{e_3}(z, t) \sqrt{c^2 - v_{e_3}^2} \sqrt{-g^{e_3}} \\
&+ \frac{1}{2} c \rho_{e_4}(z, t) \sqrt{c^2 - v_{e_4}^2} \sqrt{-g^{e_4}}. \quad (64)
\end{aligned}$$

Here we define the function space,

$$\begin{aligned}
V_{22} &= \left\{ \hat{T} = (T_{H_2}, T_{H_a^+}, T_{H_b^+}, T_e) \in W^{1,2}(\Omega \times [0, T]; \mathbb{R}^{1 \times 4}) \right. \\
&\quad \left. : \hat{T}(z, t) = T_0(z, t) \text{ on } \partial\Omega \times [0, T] \text{ and } \hat{T}(z, 0) = T_1(z) \text{ in } \Omega \right\}. \quad (65)
\end{aligned}$$

Moreover, we define the following entropy differentials

$$\begin{aligned}
dS_{H_2} &= -\hat{K}_{H_2} \frac{\rho_{H_2}}{\rho} \left(\ln \left(\frac{\rho_{H_2}}{e \rho} \right) - \omega_{H_2} \right) dx dt, \\
dS_{H_a^+} &= -\hat{K}_{H_a^+} \frac{\rho_{H_a^+}}{\rho} \left(\ln \left(\frac{\rho_{H_a^+}}{e \rho} \right) - \omega_{H_a^+} \right) dx dt,
\end{aligned}$$

$$\begin{aligned} dS_{H_b^+} &= -\hat{K}_{H_b^+} \frac{\rho_{H_b^+}}{\rho} \left(\ln \left(\frac{\rho_{H_b^+}}{e \rho} \right) - \omega_{H_b^+} \right) dxdt, \\ dS_{e_3} &= -\hat{K}_e \frac{\rho_{e_3}}{\rho} \left(\ln \left(\frac{\rho_{e_3}}{e \rho} \right) - \omega_{e_3} \right) dxdt, \\ dS_{e_4} &= -\hat{K}_e \frac{\rho_{e_4}}{\rho} \left(\ln \left(\frac{\rho_{e_4}}{e \rho} \right) - \omega_{e_4} \right) dxdt \end{aligned}$$

and the following functional

$$\begin{aligned} J_S &= \int_0^T \int_{\Omega} T_{H_2} dS_{H_2} + \int_0^T \int_{\Omega} T_{H_a^+} dS_{H_a^+} + \int_0^T \int_{\Omega} T_{H_b^+} dS_{H_b^+} \\ &\quad + \int_0^T \int_{\Omega} T_e dS_{e_3} + \int_0^T \int_{\Omega} T_e dS_{e_4}. \end{aligned} \quad (66)$$

We define also

$$\begin{aligned} S_{H_2} &= -\hat{K}_{H_2} \frac{\rho_{H_2}}{\rho} \left(\ln \left(\frac{\rho_{H_2}}{e \rho} \right) - \omega_{H_2} \right), \\ S_{H_a^+} &= -\hat{K}_{H_a^+} \frac{\rho_{H_a^+}}{\rho} \left(\ln \left(\frac{\rho_{H_a^+}}{e \rho} \right) - \omega_{H_a^+} \right), \\ S_{H_b^+} &= -\hat{K}_{H_b^+} \frac{\rho_{H_b^+}}{\rho} \left(\ln \left(\frac{\rho_{H_b^+}}{e \rho} \right) - \omega_{H_b^+} \right), \\ S_{e_3} &= -\hat{K}_e \frac{\rho_{e_3}}{\rho} \left(\ln \left(\frac{\rho_{e_3}}{e \rho} \right) - \omega_{e_3} \right), \\ S_{e_4} &= -\hat{K}_e \frac{\rho_{e_4}}{\rho} \left(\ln \left(\frac{\rho_{e_4}}{e \rho} \right) - \omega_{e_4} \right) \end{aligned}$$

and

$$E_S^M = T_{H_2} S_{H_2} + T_{H_a^+} S_{H_a^+} + T_{H_b^+} S_{H_b^+} + T_{H_e} (S_{e_3} + S_{e_4}).$$

With such results in mind, already including a term related to entropy, we define the restriction

$$\begin{aligned} &\rho_{H_2}(C_v)_{H_2} \frac{dT_{H_2}}{dt} + \rho_{H_a^+}(C_v)_{H_a^+} \frac{dT_{H_a^+}}{dt} + \rho_{H_b^+}(C_v)_{H_b^+} \frac{dT_{H_b^+}}{dt} + (C_v)_e \rho_e \frac{dT_e}{dt} \\ &= K_{H_2} \nabla^2 T_{H_2} + K_{H_a^+} \nabla^2 T_{H_a^+} + K_{H_b^+} \nabla^2 T_{H_b^+} + K_e \nabla^2 T_e \\ &\quad - P_{H_2} \operatorname{div} \mathbf{v}_{H_2} - P_{H_a^+} \operatorname{div} \mathbf{v}_{H_a^+} - P_{H_b^+} \operatorname{div} \mathbf{v}_{H_b^+} \\ &\quad - P_{e_3} \operatorname{div} \mathbf{v}_{e_3} - P_{e_4} \operatorname{div} \mathbf{v}_{e_4} + \frac{dE_S^M}{dt}. \end{aligned} \quad (67)$$

With such restrictions and definitions in mind, we define the following functionals,

$$\begin{aligned}
 J_{Aux_{30}} = & \int_0^T \int_{\Omega} E_{37}(z, t) \\
 & \times \left((C_v)_{H_2} \rho_{H_2} T_{H_2} - \frac{1}{2} c \rho_{e_1}(z, t) \sqrt{c^2 - g_{jk}^{e_1} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{e_1}} |\det X'_{H_2}| \right. \\
 & - \frac{1}{2} c \rho_{p_1}(z, t) \sqrt{c^2 - g_{jk}^{p_1} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{p_1}} |\det X'_{H_2}| \\
 & - \frac{1}{2} c \rho_{e_2}(z, t) \sqrt{c^2 - g_{jk}^{e_2} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{e_2}} |\det X'_{H_2}| \\
 & \left. - \frac{1}{2} c \rho_{p_2}(z, t) \sqrt{c^2 - g_{jk}^{p_2} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{p_2}} |\det X'_{H_2}| \right) dz dt \quad (68)
 \end{aligned}$$

$$\begin{aligned}
 J_{Aux_{31}} = & \int_0^T \int_{\Omega} E_{38}(z, t) \left((C_v)_{H_a^+} \rho_{H_a^+} T_{H_a^+} \right. \\
 & \left. - \frac{1}{2} c \rho_{p_3}(z, t) \sqrt{c^2 - g_{jk}^{p_3} \frac{\partial(X_{H_a^+})_j}{\partial t} \frac{\partial(X_{H_a^+})_k}{\partial t}} \sqrt{-g^{p_3}} |\det X'_{H_a^+}| \right) dz dt \quad (69)
 \end{aligned}$$

$$\begin{aligned}
 J_{Aux_{32}} = & \int_0^T \int_{\Omega} E_{39}(z, t) \left((C_v)_{H_b^+} \rho_{H_b^+} T_{H_b^+} \right. \\
 & \left. - \frac{1}{2} c \rho_{p_4}(z, t) \sqrt{c^2 - g_{jk}^{p_4} \frac{\partial(X_{H_b^+})_j}{\partial t} \frac{\partial(X_{H_b^+})_k}{\partial t}} \sqrt{-g^{p_4}} |\det X'_{H_b^+}| \right) dz dt \quad (70)
 \end{aligned}$$

and

$$\begin{aligned}
 J_{Aux_{33}} = & \int_0^T \int_{\Omega} E_{40}(z, t) \left((C_v)_e \rho_e T_e - \frac{1}{2} c \rho_{e_3}(z, t) \sqrt{c^2 - v_{e_3}^2} \sqrt{-g^{e_3}} \right. \\
 & \left. - \frac{1}{2} c \rho_{e_4}(z, t) \sqrt{c^2 - v_{e_4}^2} \sqrt{-g^{e_4}} \right) dz dt. \quad (71)
 \end{aligned}$$

With such results in mind, we define the restriction

$$\begin{aligned}
J_{Aux_{34}} = & \int_0^T \int_{\Omega} E_{41}(z, t) \\
& \times \left(\rho_{H_2}(C_v)_{H_2} \frac{dT_{H_2}}{dt} + \rho_{H_a^+}(C_v)_{H_a^+} \frac{dT_{H_a^+}}{dt} + \rho_{H_b^+}(C_v)_{H_b^+} \frac{dT_{H_b^+}}{dt} + (C_v)_e \frac{dT_e}{dt} \right. \\
& - K_{H_2} \nabla^2 T_{H_2} - K_{H_a^+} \nabla^2 T_{H_a^+} - K_{H_b^+} \nabla^2 T_{H_b^+} - K_e \rho_e \nabla^2 T_e \\
& + P_{H_2} \operatorname{div} \mathbf{v}_{H_2} + P_{H_a^+} \operatorname{div} \mathbf{v}_{H_a^+} + P_{H_b^+} \operatorname{div} \mathbf{v}_{H_b^+} \\
& \left. + P_{e_3} \operatorname{div} \mathbf{v}_{e_3} + P_{e_4} \operatorname{div} \mathbf{v}_{e_4} - \frac{dE_S^M}{dt} \right) dz dt
\end{aligned} \tag{72}$$

Finally, we define the following functionals,

$$\begin{aligned}
J_{Aux_{35}} = & \int_0^T \int_{\Omega} \int_{\Omega} E_{50}(z, t) (\rho_{H_2}(z, t) \\
& - \int_{\Omega} \int_{\Omega} (|\phi_{e_1}(x, y, z, t)|^2 + |\phi_{p_1}(x, y, z, t)|^2) \\
& \times \frac{|\phi_{A_1}(y, z, t)|^2}{2m_e + 2m_p} |\phi_{H_2}(z, t)|^2 \\
& - (|\phi_{e_2}(x, y, z, t)|^2 + |\phi_{p_2}(x, y, z, t)|^2) \\
& \times \frac{|\phi_{A_2}(y, z, t)|^2}{2m_e + 2m_p} |\phi_{H_2}(z, t)|^2) dx dy dz dt,
\end{aligned} \tag{73}$$

$$\begin{aligned}
J_{Aux_{36}} = & \int_0^T \int_{\Omega} \int_{\Omega} E_{60}(z, t) (\rho_{H_a^+}(z, t) \\
& - \int_{\Omega} \int_{\Omega} \left(\frac{|\phi_{p_3}(x, y, z, t)|^2}{m_p} \right) \\
& \times |\phi_{A_3}(y, z, t)|^2 |\phi_{H_a^+}(z, t)|^2) dx dy dz dt,
\end{aligned} \tag{74}$$

$$\begin{aligned}
& \stackrel{J_{A u x_{37}}}{=} \int_0^T \int_{\Omega} \int_{\Omega} E_{70}(z, t) \left(\rho_{H_b^+}(z, t) \right. \\
& \quad \left. - \int_{\Omega} \int_{\Omega} \left(\frac{|\phi_{p_4}(x, y, z, t)|^2}{m_p} \right) \right. \\
& \quad \left. \times |\phi_{A_4}(y, z, t)|^2 |\phi_{H_b^+}(z, t)|^2 \right) dx dy dz dt, \tag{75}
\end{aligned}$$

6 The main functional

Here we define

$$(\mathbf{A}, V) \in V_{23} = W^{1,2}(\Omega \times [0, T]; \mathbb{R}^3) \times W^{1,2}(\Omega \times [0, T]),$$

where $\mathbf{A}$ is a magnetic potential and V is an electric potential.

Moreover, we denote by ρ^e , m_e an electron charge and mass and ρ^p , m_p a proton charge and mass, respectively. Moreover, we define

$$K_1 = K_0 \frac{\rho_e}{m_e}$$

and $K_2 = K_0 \frac{\rho^p}{m_p}$, for an appropriate real constant $K_0 > 0$, and define the functionals

$$J : V_1 \times \cdots \times V_{23} \rightarrow \mathbb{R}$$

and

$$J_1 : V_1 \times \cdots \times V_{23} \rightarrow \mathbb{R},$$

by

$$\begin{aligned}
J = & \frac{1}{2} \int_0^T \int_{\Omega} \rho_{H_2} \mathbf{v}_{H_2} \cdot \mathbf{v}_{H_2} \, dz \, dt + \int_0^T \int_{\Omega} \frac{\rho_{e_1}}{\sqrt{1 - \frac{v_{e_1}^2}{c^2}}} \mathbf{v}_{e_1} \cdot \mathbf{v}_{H_2} \, dz \, dt \\
& + \int_0^T \int_{\Omega} \frac{\rho_{p_1}}{\sqrt{1 - \frac{v_{p_1}^2}{c^2}}} \mathbf{v}_{p_1} \cdot \mathbf{v}_{H_2} \, dz \, dt + \int_0^T \int_{\Omega} \frac{\rho_{e_2}}{\sqrt{1 - \frac{v_{e_2}^2}{c^2}}} \mathbf{v}_{e_2} \cdot \mathbf{v}_{H_2} \, dz \, dt \\
& + \int_0^T \int_{\Omega} \frac{\rho_{p_2}}{\sqrt{1 - \frac{v_{p_2}^2}{c^2}}} \mathbf{v}_{p_2} \cdot \mathbf{v}_{H_2} \, dz \, dt \\
& + \frac{1}{2} \int_0^T \int_{\Omega} \rho_{H_a^+} \mathbf{v}_{H_a^+} \cdot \mathbf{v}_{H_a^+} \, dz \, dt \\
& + \int_0^T \int_{\Omega} \frac{\rho_{p_3}}{\sqrt{1 - \frac{v_{p_3}^2}{c^2}}} \mathbf{v}_{p_3} \cdot \mathbf{v}_{H_a^+} \, dz \, dt \\
& + \frac{1}{2} \int_0^T \int_{\Omega} \rho_{H_b^+} \mathbf{v}_{H_b^+} \cdot \mathbf{v}_{H_b^+} \, dz \, dt \\
& + \int_0^T \int_{\Omega} \frac{\rho_{p_4}}{\sqrt{1 - \frac{v_{p_4}^2}{c^2}}} \mathbf{v}_{p_4} \cdot \mathbf{v}_{H_b^+} \, dz \, dt \\
& - K_1 \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{e_1} \mathbf{v}_{e_1} \cdot \mathbf{A} \, dx \, dy \, dz \, dt \\
& - K_2 \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{p_1} \mathbf{v}_{p_1} \cdot \mathbf{A} \, dx \, dy \, dz \, dt \\
& - K_1 \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{e_2} \mathbf{v}_{e_2} \cdot \mathbf{A} \, dx \, dy \, dz \, dt \\
& - K_2 \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{p_2} \mathbf{v}_{p_2} \cdot \mathbf{A} \, dx \, dy \, dz \, dt \\
& - K_2 \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{p_3} \mathbf{v}_{p_3} \cdot \mathbf{A} \, dx \, dy \, dz \, dt \\
& - K_2 \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} \hat{\rho}_{p_4} \mathbf{v}_{p_4} \cdot \mathbf{A} \, dx \, dy \, dz \, dt \\
& - K_1 \int_0^T \int_{\Omega} \rho_{e_3} \mathbf{v}_{e_3} \cdot \mathbf{A} \, dz \, dt \\
& - K_1 \int_0^T \int_{\Omega} \rho_{e_4} \mathbf{v}_{e_4} \cdot \mathbf{A} \, dz \, dt \\
& - \mathbf{g} \cdot \rho_{H_2} \mathbf{r}_{H_2} - \mathbf{g} \cdot \rho_{H_a^+} \mathbf{r}_{H_a^+} - \mathbf{g} \cdot \rho_{H_b^+} \mathbf{r}_{H_b^+} \\
& - \mathbf{g} \cdot \rho_{e_3} \mathbf{r}_{e_3} - \mathbf{g} \cdot \rho_{e_4} \mathbf{r}_{e_4}.
\end{aligned} \tag{76}$$

and, generically denoting

$$\hat{V}(\mathbf{r}_{H_a^+}, \mathbf{r}_{H_b^+}, \mathbf{r}_{e_3}, \mathbf{r}_{e_4}) = V(z, t)$$

and

$$\mathbf{B} = \text{Curl } \mathbf{A} - \mathbf{B}_0,$$

we also define

$$\begin{aligned} J_1 = & - \int_0^T \int_{\Omega} \int_{\Omega} \int_{\Omega} V(z, t) \\ & \times (K_1 \hat{\rho}_{e_1} + K_2 \hat{\rho}_{p_1} + K_1 \hat{\rho}_{e_2} + K_2 \hat{\rho}_{p_2} \\ & + K_2 \hat{\rho}_{p_3} + K_2 \hat{\rho}_{p_4} + K_1 \hat{\rho}_{e_3} + K_1 \hat{\rho}_{e_4}) \, dx \, dy \, dz \, dt \\ & + \frac{1}{8\pi} \| \text{Curl } \mathbf{A}(z, t) - \mathbf{B}_0 \|_{0,2}^2 \\ & - \frac{1}{8\pi} \left\| \nabla_z V(z, t) + \frac{1}{c} \frac{\partial \mathbf{A}(z, t)}{\partial t} \right\|_{0,2}^2 \\ & - K_2 \int_0^T \int_{\Omega} |\phi_{H_a^+}|^2 \mathbf{B}(x, t) \times \frac{\partial \mathbf{r}_{H_a^+}(x, t)}{\partial t} \cdot (\mathbf{r}_{H_a^+}(\varepsilon t, x) - (\mathbf{r}_0)_{H_a^+}(x)) \, dx dt \\ & - K_2 \int_0^T \int_{\Omega} |\phi_{H_b^+}|^2 \mathbf{B}(x, t) \times \frac{\partial \mathbf{r}_{H_b^+}(x, t)}{\partial t} \cdot (\mathbf{r}_{H_b^+}(\varepsilon t, x) - (\mathbf{r}_0)_{H_b^+}(x)) \, dx dt \\ & - K_1 \int_0^T \int_{\Omega} |\phi_{e_3}|^2 \mathbf{B}(x, t) \times \frac{\partial \mathbf{r}_{e_3}(x, t)}{\partial t} \cdot (\mathbf{r}_{e_3}(\varepsilon t, x) - (\mathbf{r}_0)_{e_3}(x)) \, dx dt \\ & - K_1 \int_0^T \int_{\Omega} |\phi_{e_4}|^2 \mathbf{B}(x, t) \times \frac{\partial \mathbf{r}_{e_4}(x, t)}{\partial t} \cdot (\mathbf{r}_{e_4}(\varepsilon t, x) - (\mathbf{r}_0)_{e_4}(x)) \, dx dt \quad (77) \end{aligned}$$

where

$$\mathbf{B} = \mathbf{B}_0 - \text{Curl } \mathbf{A}$$

is the total magnetic field, $\mathbf{B}_0$ is an external magnetic field, $\mathbf{A} = (A_1, A_2, A_3)$ is the magnetic potential,

$$\hat{V}(\mathbf{r}_{H_a^+}, \mathbf{r}_{H_b^+}, \mathbf{r}_{e_3}, \mathbf{r}_{e_4}) = V(z, t)$$

is the electric potential and

$$\mathbf{E} = \nabla V(z, t) + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t},$$

is the electric field.

The variation of $J + J_1$ in $\mathbf{r}_{H_a^+}$ which lead us to the correct Euler-Lagrange equation is defined by

$$\begin{aligned} & \hat{\delta}(J_1)_{\mathbf{r}_{H_a^+}}(\mathbf{r}_{H_a^+}, \mathbf{r}_{H_a^+}(\varepsilon t, x), \cdot, \phi_{H_a^+}, \phi_{H_b^+}, \phi_{e_3}, \phi_{e_4}, \mathbf{A}, V; \varphi) = \\ & \lim_{h \rightarrow 0} \left[J_1(\mathbf{r}_{H_a^+} + h\varphi, \mathbf{r}_{H_a^+}(\varepsilon t, x) + h\varphi, \cdot, \phi_{H_a^+}, \phi_{H_b^+}, \phi_{e_3}, \phi_{e_4}, \mathbf{A}, V) \right. \\ & \quad \left. - J_1(\mathbf{r}_{H_a^+}, \mathbf{r}_{H_a^+}(\varepsilon t, x), \cdot, \phi_{H_a^+}, \phi_{H_b^+}, \phi_{e_3}, \phi_{e_4}, \mathbf{A}, V) \right] / h, \\ & \forall \varphi \in C_c^\infty(\Omega \times [0, T]; \mathbb{R}^3). \end{aligned} \quad (78)$$

The corresponding Euler-Lagrange equation stands for

$$\begin{aligned} & \frac{\partial}{\partial t} \left(|\phi_{H_a^+}|^2 \frac{\partial \mathbf{r}_{H_a^+}(x, t)}{\partial t} \right) - K_2 \frac{\partial}{\partial t} (|\phi_{H_a^+}|^2 \mathbf{A}) \\ & \frac{\partial \hat{V}}{\partial \mathbf{r}_{H_a^+}} (K_2 |\phi_{H_a^+}^+|^2 + K_2 |\phi_{H_b^+}|^2 + K_1 |\phi_{e_3}|^2 + K_1 |\phi_{e_4}|^2) \\ & + K_2 |\phi_{H_a^+}|^2 \mathbf{B} \times \frac{\partial \mathbf{r}_{H_a^+}}{\partial t} + \mathcal{O}(\varepsilon) \\ & - \frac{1}{4\pi} \operatorname{div} \mathbf{E} \frac{\partial \hat{V}}{\partial \mathbf{r}_{H_a^+}} = 0. \end{aligned} \quad (79)$$

Considering a Maxwell equation in question and letting $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \left(|\phi_{H_a^+}|^2 \frac{\partial \mathbf{r}_{H_a^+}(x, t)}{\partial t} \right) - K_2 \frac{\partial}{\partial t} (|\phi_{H_a^+}|^2 \mathbf{A}) \\ & + K_2 |\phi_{H_a^+}|^2 \mathbf{B} \times \frac{\partial \mathbf{r}_{H_a^+}}{\partial t} = 0. \end{aligned} \quad (80)$$

We highlight the solution of the concerned system including such an Euler-Lagrange equation is computable, through the Newton's method for example, by expanding $J + J_1$ as a quadratic approximate function in φ about $(\mathbf{r}, \mathbf{r}(\varepsilon t, x))$ in each iteration.

Remark 6.1. *The variation of $J + J_1$ in $\mathbf{A}$ stands for*

$$\begin{aligned} & \operatorname{curl} (\operatorname{curl} \mathbf{A} - \mathbf{B}_0) + \frac{1}{c} \frac{\partial}{\partial t} \left(\nabla \hat{V}(\mathbf{r}_{H_a^+}, \mathbf{r}_{H_b^+}, \mathbf{r}_{e_3}, \mathbf{r}_{e_4}) + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \\ & + \mathcal{O}(\varepsilon) + 4\pi K_2 |\phi_{H_a^+}|^2 \frac{\partial \mathbf{r}_{H_a^+}}{\partial t} + 4\pi K_2 |\phi_{H_b^+}|^2 \frac{\partial \mathbf{r}_{H_b^+}}{\partial t} \\ & + 4\pi K_1 |\phi_{e_3}|^2 \frac{\partial \mathbf{r}_{e_3}}{\partial t} + 4\pi K_1 |\phi_{e_4}|^2 \frac{\partial \mathbf{r}_{e_4}}{\partial t} \\ & = 0. \end{aligned} \quad (81)$$

Letting $\varepsilon \rightarrow 0$, we have the following Maxwell equation

$$\text{curl } \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - 4\pi \tilde{\mathbf{J}} = \mathbf{0},$$

where

$$\begin{aligned} \tilde{\mathbf{J}} = & K_2 |\phi_{H_a^+}|^2 \frac{\partial \mathbf{r}_{H_a^+}}{\partial t} + K_2 |\phi_{H_b^+}|^2 \frac{\partial \mathbf{r}_{H_b^+}}{\partial t} \\ & + K_1 |\phi_{e_3}|^2 \frac{\partial \mathbf{r}_{e_3}}{\partial t} + K_1 |\phi_{e_4}|^2 \frac{\partial \mathbf{r}_{e_4}}{\partial t}. \end{aligned} \quad (82)$$

The variation of $J + J_1$ in V stands for

$$\begin{aligned} & -4\pi K_2 (|\phi_{H_a^+}|^2 + |\phi_{H_b^+}|^2) - 4\pi K_1 (|\phi_{e_3}|^2 + |\phi_{e_4}|^2) \\ & + \text{div} \left(\nabla \hat{V}(\mathbf{r}_{H_a^+}, \mathbf{r}_{H_b^+}, \mathbf{r}_{e_3}, \mathbf{r}_{e_4}) + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{0}, \end{aligned} \quad (83)$$

so that we obtain another Maxwell equation

$$\text{div } \mathbf{E} = 4\pi K_2 (|\phi_{H_a^+}|^2 + |\phi_{H_b^+}|^2) + 4\pi K_1 (|\phi_{e_3}|^2 + |\phi_{e_4}|^2)$$

Moreover, from

$$\mathbf{B} = - \text{curl } \mathbf{A} + \mathbf{B}_0$$

and assuming $\text{div}(\mathbf{B}_0) = 0$, we have got

$$\text{div } \mathbf{B} = - \text{div}(\text{curl } \mathbf{A}) = 0,$$

which is a third Maxwell equation.

Finally, the fourth Maxwell equation is obtained recalling that

$$\mathbf{E} = \nabla \hat{V}(\mathbf{r}_{H_a^+}, \mathbf{r}_{H_b^+}, \mathbf{r}_{e_3}, \mathbf{r}_{e_4}) + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t},$$

so that

$$\text{curl } \mathbf{E} = \text{curl}(\nabla V) + \frac{1}{c} \text{curl} \left(\frac{\partial \mathbf{A}}{\partial t} \right),$$

and therefore,

$$\text{curl } \mathbf{E} - \frac{1}{c} \left(\frac{\partial \text{curl } \mathbf{A}}{\partial t} \right) = \mathbf{0},$$

that is,

$$\text{curl } \mathbf{E} - \frac{1}{c} \left(\frac{\partial (-\mathbf{B} + \mathbf{B}_0)}{\partial t} \right) = \mathbf{0}.$$

Thus, if $\mathbf{B}_0$ is time independent, that is,

$$\frac{\partial \mathbf{B}_0}{\partial t} = \mathbf{0},$$

we have

$$\text{curl } \mathbf{E} + \frac{1}{c} \left(\frac{\partial \mathbf{B}}{\partial t} \right) = \mathbf{0}.$$

In summary, we have got all the four Maxwell equations as necessary conditions for a extremal point of J_3 .

The final variational formulation J_{total} for such an ionization process stands for

$$\begin{aligned} J_{total} &= J + J_1 - J_S \\ &\quad - \sum_{k=1}^8 E_{C_k} + E_{in} \\ &\quad + \sum_{k=1}^5 E_{M_k} + \sum_{k=1}^{37} J_{Aux_k}. \end{aligned} \quad (84)$$

Observe that the extremal equation

$$\frac{\partial J_{total}}{\partial \rho_{H_2}} = \mathbf{0}$$

stands for

$$\begin{aligned} &\frac{\partial J}{\partial \rho_{H_2}} - \frac{\partial J_S}{\partial \rho_{H_2}} + \frac{\partial E_{in}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux_{30}}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux_{11}}}{\partial \rho_{H_2}} \\ &\quad + \frac{\partial J_{Aux_{34}}}{\partial \rho_{H_2}} + \frac{d\hat{P}}{dt} - \hat{P} \operatorname{div} \mathbf{v}_{H_2} + E_{50}(z, t) \\ &= \mathbf{0}, \end{aligned} \quad (85)$$

where,

$$\frac{\partial J}{\partial \rho_{H_2}} = \frac{1}{2} \mathbf{v}_{H_2} \cdot \mathbf{v}_{H_2} - \mathbf{g} \cdot \mathbf{r}_{H_2}.$$

Furthermore,

$$\frac{\partial E_{in}}{\partial \rho_{H_2}} = \alpha_1 \rho_{H_2} - \beta_1 \quad (86)$$

$$\begin{aligned}\frac{\partial J_{Aux30}}{\partial \rho_{H_2}} &= E_{37}(z, t) C_{v_{H_2}} T_{H_2}, \\ \frac{\partial J_{Aux34}}{\partial \rho_{H_2}} &= -E_{41}(z, t) \mathbf{g} \cdot \mathbf{v}_{H_2} + E_{41}(z, t) \frac{dE_S^M}{dt}.\end{aligned}$$

and $\frac{\partial J_{Aux11}}{\partial \rho_{H_2}}$ is a Gâteaux differential related to the following variation,

$$\begin{aligned}& \delta J_{Aux11}(\rho_{H_2}; \varphi) \\ &= \int_0^T E_{15}(t) \left(\int_{\Omega} \varphi(y, 0) dy - \int_{\Omega} \varphi(y, t) dy \right. \\ & \quad \left. - \int_0^t \int_{\Omega} \varphi(y, \tau) \mathbf{v}_{H_2} \cdot \mathbf{n} dS d\tau \right) dt,\end{aligned}\tag{87}$$

$\forall \varphi \in C^\infty(\Omega \times [0, T])$.

Observe also that

$$P_{H_2} = \frac{d(\hat{P} \rho_{H_2})}{dt},$$

so that

$$P_{H_2} = \frac{d\hat{P}}{dt} \rho_{H_2} + \hat{P} \frac{d\rho_{H_2}}{dt}.$$

Thus,

$$\frac{d\hat{P}}{dt} = \frac{P_{H_2}}{\rho_{H_2}} - \frac{\hat{P}}{\rho_{H_2}} \frac{d\rho_{H_2}}{dt}.$$

From such last results, we may infer that

$$\begin{aligned}& \frac{\partial J}{\partial \rho_{H_2}} - \frac{\partial J_S}{\partial \rho_{H_2}} + \frac{\partial E_{in}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux30}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux11}}{\partial \rho_{H_2}} \\ & + \frac{\partial J_{Aux34}}{\partial \rho_{H_2}} + \frac{d\hat{P}}{dt} - \hat{P} \operatorname{div} \mathbf{v}_{H_2} + E_{50}(z, t) \\ &= \frac{\partial J}{\partial \rho_{H_2}} - \frac{\partial J_S}{\partial \rho_{H_2}} + \frac{\partial E_{in}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux30}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux34}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux11}}{\partial \rho_{H_2}} \\ & \quad \frac{P_{H_2}}{\rho_{H_2}} - \frac{\hat{P}}{\rho_{H_2}} \frac{d\rho_{H_2}}{dt} - \hat{P} \operatorname{div} \mathbf{v}_{H_2} + E_{50}(z, t) \\ &= \frac{\partial J}{\partial \rho_{H_2}} - \frac{\partial J_S}{\partial \rho_{H_2}} + \frac{\partial E_{in}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux30}}{\partial \rho_{H_2}} \\ & \quad + \frac{\partial J_{Aux34}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux11}}{\partial \rho_{H_2}} + \frac{P_{H_2}}{\rho_{H_2}} + E_{50}(z, t) \\ &= \mathbf{0}.\end{aligned}\tag{88}$$

Moreover, from the extremal equation

$$\frac{\partial J_{total}}{\partial \phi_{e1}} = 0,$$

we obtain

$$\begin{aligned} & -\frac{\partial \hat{\rho}_{e1}}{\partial \phi_{e1}} c \sqrt{c^2 - g_{jk}^{e1} \frac{\partial (X_{H2})_j}{\partial t} \frac{\partial (X_{H2})_k}{\partial t}} \sqrt{-g^{e1}} |\det X'_{H2}| \\ & + \frac{\partial \hat{\rho}_{e1}}{\partial \phi_{e1}} \mathbf{v}_{e1} \cdot \mathbf{v}_{H2} / \sqrt{1 - \frac{v_{e1}^2}{c^2}} \\ & - \frac{\partial \hat{\rho}_{e1}}{\partial \phi_{e1}} \mathbf{v}_{e1} \cdot \mathbf{A} + K_1 V(z, t) \frac{\partial \hat{\rho}_{e1}}{\partial \phi_{e1}} \\ & + 2E_1(y, z, t) \phi_{e1} + \frac{\partial E_{in}}{\partial \rho_{e1}} \frac{\partial \hat{\rho}_{e1}}{\partial \phi_{e1}} \\ & - E_{50}(z, t) \frac{\phi_{e1}(x, y, z, t)}{2m_e + 2m_p} |A_1(y, z, t)|^2 |\phi_{H2}(z, t)|^2 \\ & = 0, \end{aligned} \tag{89}$$

where

$$\frac{\partial \hat{\rho}_{e1}}{\partial \phi_{e1}} = \frac{\phi_{e1}}{m_e} |\phi_{A1}|^2 \rho_{H2},$$

and

$$\begin{aligned} & \frac{\partial E_{in}}{\partial \rho_{e1}} \frac{\partial \hat{\rho}_{e1}}{\partial \phi_{e1}} \\ & = 2\alpha_1 \rho_{H2} \left(\frac{|\phi_{e1}(x, y, z, t)|}{2m_e + m_p} \right) |\phi_{A1}(y, z, t)|^2 \rho_{H2} \\ & - \beta_1 \left(\frac{\phi_{e1}(x, y, z, t)}{2m_e + 2m_p} \right) |\phi_{A1}(y, z, t)|^2 \rho_{H2}. \end{aligned} \tag{90}$$

Denoting

$$E_{50}(z, t) = -\frac{P_{H2}}{\rho_{H2}} - B,$$

where

$$B = \frac{\partial J}{\partial \rho_{H2}} - \frac{\partial J_S}{\partial \rho_{H2}} + \frac{\partial E_{in}}{\partial \rho_{H2}} + \frac{\partial J_{A ux_{30}}}{\partial \rho_{H2}} + \frac{\partial J_{A ux_{34}}}{\partial \rho_{H2}} + \frac{\partial J_{A ux_{11}}}{\partial \rho_{H2}},$$

from the last previous equation, we obtain,

$$\begin{aligned}
 & -\frac{\partial \hat{\rho}_{e_1}}{\partial \phi_{e_1}} c \sqrt{c^2 - g_{jk}^{e_1} \frac{\partial (X_{H_2})_j}{\partial t} \frac{\partial (X_{H_2})_k}{\partial t}} \sqrt{-g^{e_1}} |\det X'_{H_2}| \\
 & + \frac{\partial \hat{\rho}_{e_1}}{\partial \phi_{e_1}} \mathbf{v}_{e_1} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{e_1}^2}{c^2}} \\
 & - \frac{\partial \hat{\rho}_{e_1}}{\partial \phi_{e_1}} \mathbf{v}_{e_1} \cdot \mathbf{A} + K_1 V(x, y, z, t) \frac{\partial \hat{\rho}_{e_1}}{\partial \phi_{e_1}} \\
 & + 2E_1(y, z, t) \phi_{e_1} + \frac{\partial E_{in}}{\partial \rho_{e_1}} \frac{\partial \hat{\rho}_{e_1}}{\partial \phi_{e_1}} \\
 & + \left(\frac{P_{H_2}}{\rho_{H_2}} + B \right) \frac{\phi_{e_1}(x, y, z, t)}{2m_e + 2m_p} |A_1(y, z, t)|^2 |\phi_{H_2}(z, t)|^2 \\
 & = 0.
 \end{aligned} \tag{91}$$

Multiplying both sides of this last equation by $\phi_{e_1}^*$, we have

$$\begin{aligned}
 & -\hat{\rho}_{e_1} c \sqrt{c^2 - g_{jk}^{e_1} \frac{\partial (X_{H_2})_j}{\partial t} \frac{\partial (X_{H_2})_k}{\partial t}} \sqrt{-g^{e_1}} |\det X'_{H_2}| \\
 & + \hat{\rho}_{e_1} \mathbf{v}_{e_1} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{e_1}^2}{c^2}} \\
 & - \hat{\rho}_{e_1} \mathbf{v}_{e_1} \cdot \mathbf{A} + K_1 V(z, t) \hat{\rho}_{e_1} \\
 & + 2E_1(y, z, t) |\phi_{e_1}|^2 + \frac{\partial E_{in}}{\partial \rho_{e_1}} \hat{\rho}_{e_1} \\
 & + \left(\frac{P_{H_2}}{\rho_{H_2}} + B \right) \frac{2|\phi_{e_1}(x, y, z, t)|^2}{2m_e + 2m_p} |A_1(y, z, t)|^2 |\phi_{H_2}(z, t)|^2 \\
 & = 0.
 \end{aligned} \tag{92}$$

Similarly, we may obtain

$$\begin{aligned}
 & -\hat{\rho}_{p_1} c \sqrt{c^2 - g_{jk}^{p_1} \frac{\partial (X_{H_2})_j}{\partial t} \frac{\partial (X_{H_2})_k}{\partial t}} \sqrt{-g^{p_1}} |\det X'_{H_2}| \\
 & + \hat{\rho}_{p_1} \mathbf{v}_{p_1} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{p_1}^2}{c^2}} \\
 & - \hat{\rho}_{p_1} \mathbf{v}_{p_1} \cdot \mathbf{A} + K_1 V(z, t) \hat{\rho}_{p_1} \\
 & + E_5(y, z, t) |\phi_{p_1}|^2 + \frac{\partial E_{in}}{\partial \rho_{p_1}} \hat{\rho}_{p_1} \\
 & + \left(\frac{P_{H_2}}{\rho_{H_2}} + B \right) \frac{2|\phi_{p_1}(x, y, z, t)|^2}{2m_e + 2m_p} |A_1(y, z, t)|^2 |\phi_{H_2}(z, t)|^2 \\
 & = 0,
 \end{aligned} \tag{93}$$

$$\begin{aligned}
& -\hat{\rho}_{e_2} c \sqrt{c^2 - g_{jk}^{e_2} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{e_2}} |\det X'_{H_2}| \\
& + \hat{\rho}_{e_2} \mathbf{v}_{e_2} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{e_2}^2}{c^2}} \\
& - \hat{\rho}_{e_2} \mathbf{v}_{e_2} \cdot \mathbf{A} + K_1 V(z, t) \hat{\rho}_{e_2} \\
& + 2E_1(y, z, t) |\phi_{e_2}|^2 + \frac{\partial E_{in}}{\partial \rho_{e_2}} \hat{\rho}_{e_2} \\
& + \left(\frac{P_{H_2}}{\rho_{H_2}} + B \right) \frac{2|\phi_{e_2}(x, y, z, t)|^2}{2m_e + 2m_p} |A_2(y, z, t)|^2 |\phi_{H_2}(z, t)|^2 \\
& = 0
\end{aligned} \tag{94}$$

and

$$\begin{aligned}
& -\hat{\rho}_{p_2} c \sqrt{c^2 - g_{jk}^{p_2} \frac{\partial(X_{H_2})_j}{\partial t} \frac{\partial(X_{H_2})_k}{\partial t}} \sqrt{-g^{p_2}} |\det X'_{H_2}| \\
& + \hat{\rho}_{p_2} \mathbf{v}_{p_2} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{p_2}^2}{c^2}} \\
& - \hat{\rho}_{p_2} \mathbf{v}_{p_2} \cdot \mathbf{A} + K_1 V(z, t) \hat{\rho}_{p_2} \\
& + E_7(y, z, t) |\phi_{p_2}|^2 + \frac{\partial E_{in}}{\partial \rho_{p_2}} \hat{\rho}_{p_2} \\
& + \left(\frac{P_{H_2}}{\rho_{H_2}} + B \right) \frac{2|\phi_{p_2}(x, y, z, t)|^2}{2m_e + 2m_p} |A_2(y, z, t)|^2 |\phi_{H_2}(z, t)|^2 \\
& = 0.
\end{aligned} \tag{95}$$

where

$$\begin{aligned}
& \frac{\partial E_{in}}{\partial \rho_{e_2}} \frac{\partial \hat{\rho}_{e_2}}{\partial \phi_{e_2}} \\
& = 2\alpha_1 \rho_{H_2} \left(\frac{|\phi_{e_2}(x, y, z, t)|}{2m_e + 2m_p} \right) |\phi_{A_2}(y, z, t)|^2 \rho_{H_2} \\
& - \beta_1 \left(\frac{\phi_{e_2}(x, y, z, t)}{2m_e + 2m_p} \right) |\phi_{A_2}(y, z, t)|^2 \rho_{H_2},
\end{aligned} \tag{96}$$

$$\begin{aligned}
& \frac{\partial E_{in}}{\partial \rho_{p_1}} \frac{\partial \hat{\rho}_{p_1}}{\partial \phi_{p_1}} \\
& = 2\alpha_1 \rho_{H_2} \left(\frac{|\phi_{p_1}(x, y, z, t)|}{2m_e + 2m_p} \right) |\phi_{A_1}(y, z, t)|^2 \rho_{H_2} \\
& - \beta_1 \left(\frac{\phi_{p_1}(x, y, z, t)}{2m_e + 2m_p} \right) |\phi_{A_1}(y, z, t)|^2 \rho_{H_2},
\end{aligned} \tag{97}$$

and

$$\begin{aligned}
& \frac{\partial E_{in}}{\partial \rho_{p_2}} \frac{\partial \hat{\rho}_{p_2}}{\partial \phi_{p_2}} \\
&= 2\alpha_1 \rho_{H_2} \left(\frac{|\phi_{p_2}(x, y, z, t)|}{2m_e + 2m_p} \right) |\phi_{A_2}(y, z, t)|^2 \rho_{H_2} \\
&\quad - \beta_1 \left(\frac{\phi_{p_2}(x, y, z, t)}{2m_e + 2m_p} \right) |\phi_{A_2}(y, z, t)|^2 \rho_{H_2}.
\end{aligned} \tag{98}$$

Summing up the last previous equations and integrating them in x, y and recalling that

$$\begin{aligned}
& (C_v)_{H_2} \rho_{H_2} T_{H_2} \\
&= \frac{1}{2} c \rho_{e_1}(z, t) \sqrt{c^2 - g_{jk}^{e_1} \frac{\partial (X_{H_2})_j}{\partial t} \frac{\partial (X_{H_2})_k}{\partial t}} \sqrt{-g^{e_1}} |\det X'_{H_2}| \\
&\quad + \frac{1}{2} c \rho_{p_1}(z, t) \sqrt{c^2 - g_{jk}^{p_1} \frac{\partial (X_{H_2})_j}{\partial t} \frac{\partial (X_{H_2})_k}{\partial t}} \sqrt{-g^{p_1}} |\det X'_{H_2}| \\
&\quad + \frac{1}{2} c \rho_{e_2}(z, t) \sqrt{c^2 - g_{jk}^{e_2} \frac{\partial (X_{H_2})_j}{\partial t} \frac{\partial (X_{H_2})_k}{\partial t}} \sqrt{-g^{e_2}} |\det X'_{H_2}| \\
&\quad + \frac{1}{2} c \rho_{p_2}(z, t) \sqrt{c^2 - g_{jk}^{p_2} \frac{\partial (X_{H_2})_j}{\partial t} \frac{\partial (X_{H_2})_k}{\partial t}} \sqrt{-g^{p_2}} |\det X'_{H_2}|,
\end{aligned} \tag{99}$$

we obtain

$$-(C_v)_{H_2} \rho_{H_2} T_{H_2} + C_1 + \left(\frac{P_{H_2}}{\rho_{H_2}} + B \right) \rho_{H_2} = 0, \tag{100}$$

where,

$$B = \frac{\partial J}{\partial \rho_{H_2}} - \frac{\partial J_S}{\partial \rho_{H_2}} + \frac{\partial E_{in}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux_{30}}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux_{34}}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux_{11}}}{\partial \rho_{H_2}},$$

so that

$$\begin{aligned}
B &= \frac{1}{2} \mathbf{v}_{H_2} \cdot \mathbf{v}_{H_2} + g(X_{H_2})_3 + \alpha_1 \rho_{H_2} - \beta_1 + \hat{K}_{H_2} T_{H_2} \left(\ln \left(\frac{\rho_{H_2}}{\rho} \right) - \omega_{H_e} \right) \\
&\quad + \frac{\partial J_{Aux_{30}}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux_{34}}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux_{11}}}{\partial \rho_{H_2}},
\end{aligned} \tag{101}$$

and

$$\begin{aligned}
C_1 = & \hat{\rho}_{e_1} \mathbf{v}_{e_1} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{e_1}^2}{c^2}} \\
& - \hat{\rho}_{e_1} \mathbf{v}_{e_1} \cdot \mathbf{A} + K_1 V(z, t) \hat{\rho}_{e_1} \\
& + 2E_1(y, z, t) |\phi_{e_1}|^2 + \frac{\partial E_{in}}{\partial \rho_{e_1}} \hat{\rho}_{e_1} \\
& + \hat{\rho}_{p_1} \mathbf{v}_{p_1} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{p_1}^2}{c^2}} \\
& - \hat{\rho}_{p_1} \mathbf{v}_{p_1} \cdot \mathbf{A} + K_1 V(z, t) \hat{\rho}_{p_1} \\
& + E_5(y, z, t) |\phi_{p_1}|^2 + \frac{\partial E_{in}}{\partial \rho_{p_1}} \hat{\rho}_{p_1} + \hat{\rho}_{e_2} \mathbf{v}_{e_2} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{e_2}^2}{c^2}} \\
& - \hat{\rho}_{e_2} \mathbf{v}_{e_2} \cdot \mathbf{A} + K_1 V(z, t) \hat{\rho}_{e_2} \\
& + 2E_1(y, z, t) |\phi_{e_2}|^2 + \frac{\partial E_{in}}{\partial \rho_{e_2}} \hat{\rho}_{e_2} \\
& + \hat{\rho}_{p_2} \mathbf{v}_{p_2} \cdot \mathbf{v}_{H_2} / \sqrt{1 - \frac{v_{p_2}^2}{c^2}} \\
& - \hat{\rho}_{p_2} \mathbf{v}_{p_2} \cdot \mathbf{A} + K_1 V(z, t) \hat{\rho}_{p_2} \\
& + E_7(y, z, t) |\phi_{p_2}|^2 + \frac{\partial E_{in}}{\partial \rho_{p_2}} \hat{\rho}_{p_2}.
\end{aligned} \tag{102}$$

Now denoting

$$C_3 = \frac{\partial J_{Aux30}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux34}}{\partial \rho_{H_2}} + \frac{\partial J_{Aux11}}{\partial \rho_{H_2}},$$

we obtain

$$B = \frac{1}{2} \mathbf{v}_{H_2} \cdot \mathbf{v}_{H_2} + g(X_{H_2})_3 + \alpha_1 \rho_{H_2} - \beta_1 + \hat{K}_{H_2} \frac{T_{H_2}}{\rho} \left(\ln \left(\frac{\rho_{H_2}}{\rho} \right) - \omega_{H_e} \right) + C_3.$$

From such expressions for C_1 , C_3 and from (100), we have got

$$\begin{aligned}
& \frac{P_{H_2}}{\rho_{H_2}} + \frac{1}{2} \mathbf{v}_{H_2} \cdot \mathbf{v}_{H_2} + g(X_{H_2})_3 \\
& + \alpha_1 \rho_{H_2} - \beta_1 + \hat{K}_{H_2} \frac{T_{H_2}}{\rho} \left(\ln \left(\frac{\rho_{H_2}}{\rho} \right) - \omega_{H_e} \right) + C_3 \\
= & (C_v)_{H_2} T_{H_2} - \frac{C_1}{\rho_{H_2}}.
\end{aligned} \tag{103}$$

Thus, we have obtained the following equation

$$\begin{aligned}
 & P_{H_2} + \frac{1}{2} \rho_{H_2} \mathbf{v}_{H_2} \cdot \mathbf{v}_{H_2} + g \rho_{H_2} (X_{H_2})_3 \\
 = & C_{v_{H_2}} \rho_{H_2} T_{H_2} - \alpha_1 \rho_{H_2}^2 + \left(\beta_1 + \hat{K}_{H_2} T_{H_2} \frac{\omega_{H_e}}{\rho} \right) \rho_{H_e} \\
 & - \hat{K}_{H_2} T_{H_2} \left(\ln \left(\frac{\rho_{H_2}}{\rho} \right) \right) \frac{\rho_{H_e}}{\rho} - C_1 - C_3 \rho_{H_2}.
 \end{aligned} \tag{104}$$

In summary, we have obtained the partial approximate Bernoulli-interacting-gas equation,

$$\begin{aligned}
 & P_{H_2} + \frac{\rho_{H_2}}{2} \mathbf{v}_{H_2} \cdot \mathbf{v}_{H_2} + \rho_{H_2} g (X_{H_2})_3 \\
 = & K_3 \rho_{H_2} T_{H_2} - \alpha_1 \rho_{H_2}^2 + \left(\beta_1 + \hat{K}_{H_2} T_{H_2} \frac{\omega_{H_e}}{\rho} \right) \rho_{H_e} \\
 & - \hat{K}_{H_2} T_{H_2} \left(\ln \left(\frac{\rho_{H_2}}{\rho} \right) \right) \frac{\rho_{H_e}}{\rho} + D_1,
 \end{aligned} \tag{105}$$

for an appropriate function D_1 and an appropriate $K_3 > 0$ obtained through equation (104).

Moreover, from the variation of J_{total} in $\mathbf{v}_{H_2}$, we obtain

$$\rho_{H_2} \mathbf{v}_{H_2} + E_{17} + \nabla(\hat{P} \rho_{H_2}) + C_5 = \mathbf{0},$$

where

$$\begin{aligned}
 C_5 = & \frac{\rho_{e_1}}{\sqrt{1 - \frac{v_{e_1}^2}{c^2}}} \mathbf{v}_{e_1} + \frac{\rho_{e_2}}{\sqrt{1 - \frac{v_{e_2}^2}{c^2}}} \mathbf{v}_{e_2} \\
 & \frac{\rho_{p_1}}{\sqrt{1 - \frac{v_{p_1}^2}{c^2}}} \mathbf{v}_{p_1} + \frac{\rho_{p_2}}{\sqrt{1 - \frac{v_{p_2}^2}{c^2}}} \mathbf{v}_{p_2} \\
 & + \frac{\partial J_{Aux_{34}}}{\partial \mathbf{v}_{H_2}} + \frac{\partial J_{Aux_{11}}}{\partial \mathbf{v}_{H_2}},
 \end{aligned} \tag{106}$$

$$-\frac{\partial J_{Aux_{34}}}{\partial \mathbf{v}_{H_2}} = -\nabla_z (E_{41}(z, t) P_{H_2}),$$

and $\frac{\partial J_{Aux_{11}}}{\partial \rho_{H_2}}$ is related to the following variation,

$$\delta J_{Aux_{11}}(\mathbf{v}_{H_2}; \varphi) = \int_0^T E_{15}(t) \left(- \int_0^t \int_{\Omega} \rho_{H_2} \varphi(y, \tau) \cdot \mathbf{n} dS d\tau \right) dt,$$

$\forall \varphi \in C^\infty(\Omega \times [0, T]; \mathbb{R}^3)$, which just generates a boundary condition for $E_{15}(t)$.

The variation of J_{total} in $\mathbf{r}_{H_2}$, stands for

$$-\rho_{H_2} \mathbf{g} + \frac{dE_{17}}{dt} + C_7 = 0,$$

where

$$C_7 = \frac{\partial J_{Aux_{30}}}{\partial \mathbf{r}_{H_2}} + \frac{\partial \left(\sum_{j=1}^4 E_{C_j} \right)}{\partial \mathbf{r}_{H_2}},$$

where such explicit expressions may be obtained similarly as above indicated.

Joining the pieces we have got

$$\rho_{H_2} \mathbf{g} + \frac{d(\rho_{H_2} \mathbf{v}_{H_2})}{dt} + \frac{d(\nabla(\hat{P} \rho_{H_2}))}{dt} - \frac{dC_5}{dt} - C_7 = 0,$$

so that

$$\frac{\partial(\rho_{H_2} \mathbf{v}_{H_2})}{\partial t} + \frac{\partial(\rho_{H_2} \mathbf{v}_{H_2})}{\partial (X_{H_2})_j} \frac{\partial (X_{H_2})_j}{\partial t} + \nabla P_{H_2} + \rho_{H_2} \mathbf{g} = \frac{dC_5}{dt} + C_7 \equiv C_9.$$

This just an approximate version of the well known Euler system.

Moreover, from the variation of J_{total} in $\hat{P}$, provide us,

$$\begin{aligned} & \frac{d\rho}{dt} + \rho_{H_2} \operatorname{div} \mathbf{v}_{H_2} + \rho_{H_a^+} \operatorname{div} \mathbf{v}_{H_a^+} \\ & + \rho_{H_b^+} \operatorname{div} \mathbf{v}_{H_b^+} + \rho_{e_3} \operatorname{div} \mathbf{v}_{e_3} + \rho_{e_4} \operatorname{div} \mathbf{v}_{e_4} + C_8 = 0, \end{aligned} \quad (107)$$

where

$$C_8 = \frac{\partial J_{Aux_{14}}}{\partial P} \frac{\partial P}{\partial \hat{P}}.$$

Hence, through the Eulerian identification

$$(X_{H_2})_k(y, t) \approx x_k, \quad \forall k \in \{1, 2, 3\},$$

we have obtained the following macroscopic system,

1. Euler type Equation:

$$\frac{\partial(\rho_{H_2} \mathbf{v}_{H_2})}{\partial t} + \frac{\partial(\rho_{H_2} \mathbf{v}_{H_2})}{\partial x_j} (\mathbf{v}_{H_2})_j + \nabla P_{H_2} + \rho_{H_2} \mathbf{g} = C_9,$$

2. Continuity type equation:

$$\begin{aligned} & \frac{d\rho}{dt} + \rho_{H_2} \operatorname{div} \mathbf{v}_{H_2} + \rho_{H_a^+} \operatorname{div} \mathbf{v}_{H_a^+} \\ & + \rho_{H_b^+} \operatorname{div} \mathbf{v}_{H_b^+} + \rho_{e_3} \operatorname{div} \mathbf{v}_{e_3} + \rho_{e_4} \operatorname{div} \mathbf{v}_{e_4} + C_8 = 0, \end{aligned} \quad (108)$$

3. A Temperature type equation:

$$\begin{aligned} & \rho_{H_2}(C_v)_{H_2} \frac{dT_{H_2}}{dt} + \rho_{H_a^+}(C_v)_{H_a^+} \frac{dT_{H_a^+}}{dt} \\ & + \rho_{H_b^+}(C_v)_{H_b^+} \frac{dT_{H_b^+}}{dt} + \rho_e(C_v)_e \frac{dT_e}{dt} \\ = & K_{H_2} \nabla^2 T_{H_2} + K_{H_a^+} \nabla^2 T_{H_a^+} + K_{H_b^+} \nabla^2 T_{H_b^+} + K_e \nabla^2 T_e \\ & - P_{H_2} \operatorname{div} \mathbf{v}_{H_2} - P_{H_a^+} \operatorname{div} \mathbf{v}_{H_a^+} - P_{H_b^+} \operatorname{div} \mathbf{v}_{H_b^+} \\ & - P_{e_3} \operatorname{div} \mathbf{v}_{e_3} - P_{e_4} \operatorname{div} \mathbf{v}_{e_4} + \frac{dE_S^M}{dt}. \end{aligned} \quad (109)$$

4. A Bernoulli-interacting-gas type equation:

$$\begin{aligned} & P_{H_2} + \frac{\rho_{H_2}}{2} \mathbf{v}_{H_2} \cdot \mathbf{v}_{H_2} + \rho_{H_2} g(X_{H_2})_3 \\ = & K_3 \rho_{H_2} T_{H_2} - \alpha_1 \rho_{H_2}^2 + \left(\beta_1 + \hat{K}_{H_2} T_{H_2} \frac{\omega_{H_e}}{\rho} \right) \rho_{H_e} \\ & - \hat{K}_{H_2} T_{H_2} \left(\ln \left(\frac{\rho_{H_2}}{\rho} \right) \right) \frac{\rho_{H_e}}{\rho} + D_1, \end{aligned} \quad (110)$$

for an appropriate function D_1 and an appropriate $K_3 > 0$.

Remark 6.2. We highlight the functions C_5, C_7, C_8 and C_9 may be exactly obtained from the exact concerning variations of J_{total} .

Finally, we also emphasize similar results may be obtained for $P_{H_a^+}, P_{H_b^+}, P_{e_3}$ and P_{e_4} .

The objective of this section is complete.

7 Conclusion

In this article, we have developed a variational formulation for a hydrogen molecule ionization chemical reaction.

In summary, we have obtained a functional such that the variation comprises a set of partial differential equations intended to mathematically model such an ionization process.

Finally, we highlight our main results include approximate Bernoullian-interacting-gas equations, which in some sense, make the role of the standard ideal gas and related equations in the contemporary literature in fluid mechanics.

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**HAMILTON-JACOBI TREATMENT OF THE SCALAR FIELD
COUPLED TO TWO FERMIONS**

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Abstract

The constrained field system, the scalar field coupled to two flavours of fermions through Yukawa couplings, is treated by using the Hamilton-Jacobi approach. The equations of motion are obtained as total differential equations in many variables. The equations of motion are in exact agreement with those equations obtained using Dirac's method. Due to the Hamilton-Jacobi quantization, the path integral of the scalar field coupled to two fermions is obtained directly as an integration over the canonical phase space without using any gauge fixing condition.

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1 Introduction

Investigation the Hamiltonian treatment of constrained systems was initiated by Dirac [1], which is the most common method. Considering the primary constraints first is the main feature of his method. All constraints are obtained using consistency conditions. Besides, he showed that the number of degrees of freedom of the dynamical system can be reduced. Hence, the equations of motion of constrained system are obtained in terms of arbitrary parameters.

The canonical method, or Güler's method, developed Hamilton-Jacobi approach [2-3] to investigate constrained systems. In Refs. [4, 5], the equivalent Lagrangian method is used to obtain the set of Hamilton-Jacobi partial differential equations (HJPDE). In the Hamilton-Jacobi treatment of constrained systems [6-9]: (i) the distinction between first- and second-class constraints is unnecessary (ii) the equations of motion are obtained as total differential equations in many variables (iii) the investigation of the integrability conditions is required (iv) that gauge fixing is not necessary.

Quantization of the constrained Hamiltonian systems can be achieved by using several methods which are operator methods [1, 10], by path integral quantization [11-13], and based on the canonical method [14], which called the Hamilton-Jacobi quantization [15-20].

We investigate in the present work the scalar field coupled to two flavours of fermions through Yukawa couplings by using both Hamilton-Jacobi formulation and Dirac's method. Note that, Yukawa couplings are named after Hideki Yukawa interacted a scalar field φ with a Dirac field ψ of the type, which can be used to describe the strong nuclear force between nucleons, mediated by pions, and used in the Standard Model to describe the coupling between the Higgs field and massless quark and electron fields. Through spontaneous symmetry breaking, the fermions acquire a mass proportional to the vacuum expectation value of the Higgs field. The path integral quantization of the scalar field coupled to two flavours of fermions will be also quantize by using the Hamilton-Jacobi quantization.

In this paper both methods, Dirac's and Güler's are used to tackle the system of the scalar field coupled to two flavours of fermions through Yukawa couplings. This paper is arranged as follows: Hamilton-Jacobi approach is presented in Sec.2. Dirac's method is used in Sec.3. Hamilton-

Jacobi method is applied in sec.4 and the quantization of the system in Sec.5. The paper closes with a conclusion in Sec. 6.

2 Hamilton-Jacobi approach

The Hamilton-Jacobi approach can be introduced as follows:
If the rank of the Hess matrix is:

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad (1)$$

is $(n - r)$, $r < n$, then the standard definition of a linear momenta is:

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \dots, n - r, \quad (2)$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \dots, n, \quad (3)$$

enables us to solve Eq.(2) for $\dot{q}_a$ as

$$\dot{q}_b = \omega_b(q_i, \dot{q}_\mu, p_a). \quad (4)$$

Substituting Eq.(4), into Eq.(3), we obtain the constraints as

$$H'_\mu = p_\mu + H_\mu(\tau, q_i, p_a) = 0, \quad (5)$$

where

$$H_\mu = - \frac{\partial L}{\partial \dot{q}_\mu} \Big|_{\dot{q}_a \equiv \omega_a}. \quad (6)$$

The usual Hamiltonian H_0 is defined as:

$$H_0 = -L + p_a \dot{\omega}_a - \dot{q}_\mu H_\mu. \quad (7)$$

Like functions H_μ , the function H_0 is not an explicit function of the velocities $\dot{q}_\nu$. Therefore, the Hamilton-Jacobi function $S(\tau, q_i)$ should satisfy the following set of Hamilton-Jacobi partial differential equations (HJPDE) simultaneously for an extremum of the function:

$$H'_\alpha \left(t_\beta, q_\alpha, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial t_0} \right) = 0, \quad (8)$$

where

$$\alpha, \beta = 0, n-r+1, \dots, n; \quad a = 1, 2, \dots, n-r, \text{ and}$$

$$H'_\alpha = p_\alpha + H_\alpha. \quad (9)$$

The canonical equations of motion are given as total differential equations in variables t_β ,

$$dq_p = \frac{\partial H'_\alpha}{\partial p_p} dt_\alpha, \quad p = 0, 1, \dots, n; \quad \alpha = 0, n-r+1, \dots, n, \quad (10)$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad a = 1, \dots, n-r, \quad (11)$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \quad \alpha = 0, n-r+1, \dots, n, \quad (12)$$

$$dZ = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_\alpha} dt_\alpha \right), \quad (13)$$

where

$$Z \equiv S(t_\alpha, q_a), \quad (14)$$

being the action. Thus, the analysis of a constrained system is reduced to solve equations (10-12) with constraints

$$H'_\alpha(t_\beta, q_a, P_i) = 0, \quad \alpha, \beta = 0, n-r+1, \dots, n. \quad (15)$$

Since the equations above are total differential equations, integrability conditions should be checked. These equations of motion are integrable [3-5,8] if and only if the variations of H'_α vanish identically as:

$$\delta H'_\alpha = 0. \quad (16)$$

If they do not vanish identically, then we consider them as new constraints. This procedure is repeated until a complete system is obtained.

2.1 Hamilton-Jacobi Quantization

Path integral formulation of the constrained systems is studied in Refs. [14-20]. One has to consider a singular Lagrangian, as seen in previous section, for computing the Hamilton-Jacobi path integral. The canonical Hamiltonian H_0 defined in Eq. (6), and the set of HJPDE is expressed in Eq. (7). As we define

$$p_\beta = \frac{\partial S[q_a; x_\alpha]}{\partial x_\beta}, \quad (17)$$

and

$$p_a = \frac{\partial S[q_a; x_\alpha]}{\partial q_a}, \quad (18)$$

with $x_0 = t$ and S being the action. The total differential equations given in (10-13) are integrable if (16) are hold [21]. If conditions (16) are not satisfied identically, one considers them as new constraints and again consider their variations.

Thus, repeating this procedure one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide to obtain the set of canonical phase space coordinates (q_a, p_a) as functions of t_α , besides the canonical action integral is obtained in terms of the canonical coordinates. H'_α can be interpreted as the infinitesimal generator of canonical transformations given by parameters t_α . In this case path integral representation may be written as

$$\langle Out | S | In \rangle = \int \prod_{a=1}^{n-p} dq^a dp^a \exp \left\{ \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_\alpha} \right) dt_\alpha \right\},$$

$$a = 1, \dots, n-p, \quad \alpha = 0, n-p+1, \dots, n. \quad (19)$$

In fact, this path integral is an integration over the canonical phase-space coordinates (q^a, p^a) .

3 Dirac's method

In this section, we treat the constrained field system by Dirac's method [22].

Consider one loop order the self-energy for the scalar field φ , with a mass m , coupled to two flavours of fermions, with masses m_1 and m_2 , due to Yukawa couplings

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2 \varphi^2 - \frac{1}{6}\lambda \varphi^3 + \sum_i \bar{\psi}_{(i)}(i\gamma^\mu \partial_\mu - m_i)\psi_{(i)} - g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad \mu = 0, 1, 2, 3, \quad (20)$$

where λ is parameter and g constant, φ , $\psi_{(i)}$, and $\bar{\psi}_{(i)}$ are odd ones. We are adopting the Minkowski metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

The Lagrangian function (20) is singular, since the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad (21)$$

is one.

The generalized momenta (2,3)

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \partial^0 \varphi, \quad (22)$$

$$p_{(i)} = \frac{\partial L}{\partial \dot{\psi}_{(i)}} = i\bar{\psi}_{(i)}\gamma^0 = -H_{(i)}, \quad i = 1, 2, \quad (23)$$

$$\bar{p}_{(i)} = \frac{\partial L}{\partial \dot{\bar{\psi}}_{(i)}} = 0 = -\bar{H}_{(i)}. \quad (24)$$

Where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual $\psi_{(i)}$ as a column vector and $\bar{\psi}_{(i)}$ as a row vector implies that $p_{(i)}$ will be a row vector while $\bar{p}_{(i)}$ will be a column vector.

Since the rank of the Hess matrix is one, one may solve (22) for $\partial^0 \varphi$ as:

$$\partial^0 \varphi = p_\varphi \equiv \omega. \quad (25)$$

The usual Hamiltonian H_0 is given as:

$$H_0 = -L + \omega p_\varphi + \partial_0 \psi_{(i)} p_{(i)} \Big|_{p_{(i)} = -H_{(i)}} + \partial_0 \bar{\psi}_{(i)} \bar{p}_{(i)} \Big|_{\bar{p}_{(i)} = -\bar{H}_{(i)}}, \quad (26)$$

or

$$H_0 = \frac{1}{2}(p^2_\varphi - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad a = 1, 2, 3. \quad (27)$$

Eqs. (23), and (24) lead to the primary constraints

$$H'_{(i)} = p_{(i)} + H_{(i)} = p_{(i)} - i\bar{\psi}_{(i)}\gamma^0 = 0, \quad (28)$$

and

$$\bar{H}'_{(i)} = \bar{p}_{(i)} + \bar{H}_{(i)} = \bar{p}_{(i)} = 0, \quad (29)$$

respectively. These constraints lead to the total Hamiltonian

$$H_T = H_0 + \lambda_{(i)}H'_{(i)} + \bar{\lambda}_{(i)}\bar{H}'_{(i)}, \quad (30)$$

or

$$H_T = \frac{1}{2}(p^2_\varphi - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) + \lambda_{(i)}(p_{(i)} - i\bar{\psi}_{(i)}\gamma^0) + \bar{\lambda}_{(i)}\bar{p}_{(i)}. \quad (31)$$

According to Dirac's method, the time derivative of the primary constraints should be zero, that is

$$\dot{H}'_{(1)} = \{H'_{(1)}, H_T\} = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi\bar{\psi}_{(2)} - i\bar{\lambda}_{(1)}\gamma^0 \approx 0, \quad (32)$$

$$\dot{H}'_{(2)} = \{H'_{(2)}, H_T\} = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi\bar{\psi}_{(1)} - i\bar{\lambda}_{(2)}\gamma^0 \approx 0, \quad (33)$$

$$\dot{\bar{H}}'_{(1)} = \{\bar{H}'_{(1)}, H_T\} = -(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)} - i\gamma^0 \lambda_{(1)} \approx 0, \quad (34)$$

$$\dot{\bar{H}}'_{(2)} = \{\bar{H}'_{(2)}, H_T\} = -(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)} - i\gamma^0 \lambda_{(2)} \approx 0. \quad (35)$$

Eqs. (32-35) fix the multipliers $\bar{\lambda}_{(1)}$, $\bar{\lambda}_{(2)}$, $\lambda_{(1)}$, and $\lambda_{(2)}$ respectively as

$$i\bar{\lambda}_{(1)}\gamma^0 = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi\bar{\psi}_{(2)}, \quad (36)$$

$$i\bar{\lambda}_{(2)}\gamma^0 = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi\bar{\psi}_{(1)}, \quad (37)$$

$$i\gamma^0 \lambda_{(1)} = -(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi \psi_{(2)}, \quad (38)$$

$$i\gamma^0 \lambda_{(2)} = -(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi \psi_{(1)}. \quad (39)$$

Multiplying Eqs. (36) and (37) from the right and Eqs. (38) and (39) from the left by $-i\gamma^0$, we obtain:

$$\bar{\lambda}_{(1)} = \bar{\psi}_{(1)} (\overleftarrow{\partial}_a \gamma^a - im_1) \gamma^0 - ig\varphi \bar{\psi}_{(2)} \gamma^0, \quad (40)$$

$$\bar{\lambda}_{(2)} = \bar{\psi}_{(2)} (\overleftarrow{\partial}_a \gamma^a - im_2) \gamma^0 - ig\varphi \bar{\psi}_{(1)} \gamma^0, \quad (41)$$

$$\lambda_{(1)} = -\gamma^0 (\gamma^a \partial_a + im_1)\psi_{(1)} - ig\varphi \gamma^0 \psi_{(2)}, \quad (42)$$

$$\lambda_{(2)} = -\gamma^0 (\gamma^a \partial_a + im_2)\psi_{(2)} - ig\varphi \gamma^0 \psi_{(1)}. \quad (43)$$

There are no secondary constraints. Taking suitable linear combinations of constraints, one has to find all numbers of second-class ones, there are

$$\Phi_i = H'_{(i)} = p_{(i)} - i\bar{\psi}_{(i)} \gamma^0, \quad i = 1, 2, \quad (44)$$

$$\Phi_3 = \bar{H}'_{(1)} = \bar{p}_{(1)}, \quad (45)$$

and

$$\Phi_4 = \bar{H}'_{(2)} = \bar{p}_{(2)}. \quad (46)$$

The equations of motion are read as

$$\dot{\varphi} = \{\varphi, H_T\} = p_\varphi, \quad (47)$$

$$\dot{\psi}_{(i)} = \{\psi_{(i)}, H_T\} = \lambda_{(i)}, \quad (48)$$

$$\dot{\bar{\psi}}_{(i)} = \{\bar{\psi}_{(i)}, H_T\} = \bar{\lambda}_{(i)}, \quad (49)$$

$$\dot{p}_\varphi = \{p_\varphi, H_T\} = m^2\varphi + \frac{1}{2}\lambda\varphi^2 + g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad (50)$$

$$\dot{p}_{(1)} = \{p_{(1)}, H_T\} = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) + g\varphi\bar{\psi}_{(2)}, \quad (51)$$

$$\dot{p}_{(2)} = \{p_{(2)}, H_T\} = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) + g\varphi\bar{\psi}_{(1)}, \quad (52)$$

$$\dot{\bar{p}}_{(1)} = \{\bar{p}_{(1)}, H_T\} = -(i\gamma^a\partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)} - i\gamma^0\lambda_{(1)}, \quad (53)$$

$$\dot{\bar{p}}_{(2)} = \{\bar{p}_{(2)}, H_T\} = -(i\gamma^a\partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)} - i\gamma^0\lambda_{(2)}. \quad (54)$$

Differentiate Eq. (47) with respect to time, and substituting from Eq. (50), we get:

$$\ddot{\varphi} - m^2\varphi - \frac{1}{2}\lambda\varphi^2 - g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) = 0. \quad (55)$$

Substituting from Eqs. (42) and (43) into Eqs. (48), (53) and (54), we get:

$$(i\gamma^\mu\partial_\mu - m_1)\psi_{(1)} - g\varphi\psi_{(2)} = 0, \quad (56)$$

$$(i\gamma^\mu\partial_\mu - m_2)\psi_{(2)} - g\varphi\psi_{(1)} = 0, \quad (57)$$

$$\dot{\bar{p}}_{(i)} = 0, \quad i = 1, 2. \quad (58)$$

From Eqs. (40) and (41) into Eq. (49), we have

$$\partial_0\bar{\psi}_{(1)} i\gamma^0 - \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) - g\varphi\bar{\psi}_{(2)} = 0, \quad (59)$$

$$\partial_0\bar{\psi}_{(2)} i\gamma^0 - \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) - g\varphi\bar{\psi}_{(1)} = 0. \quad (60)$$

In the following section the same system will be discussed by using Hamilton-Jacobi approach.

4 Hamilton-Jacobi method

In this section, the constrained system is going now to be tackled by using Hamilton-Jacobi treatment [23], which solve the gauge fixing problem naturally.

The set of (HJPDE) (8) read as

$$H'_0 = p_0 + H_0 = p_0 + \frac{1}{2}(p^2_\varphi - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad (61)$$

$$H'_{(i)} = p_{(i)} + H_{(i)} = p_{(i)} - i\bar{\psi}_{(i)}\gamma^0 = 0, \quad (62)$$

$$\bar{H}'_{(i)} = \bar{p}_{(i)} + \bar{H}_{(i)} = \bar{p}_{(i)} = 0. \quad (63)$$

Therefore, the total differential equations for the characteristic (10), (11) and (12) are:

$$d\varphi = p_\varphi d\tau, \quad (64)$$

$$d\psi_{(i)} = d\psi_{(i)}, \quad (65)$$

$$d\bar{\psi}_{(i)} = d\bar{\psi}_{(i)}, \quad (66)$$

$$dp_\varphi = \left[m^2 \varphi + \frac{1}{2}\lambda \varphi^2 + g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) \right] d\tau, \quad (67)$$

$$dp_{(1)} = \left[\bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi \bar{\psi}_{(2)} \right] d\tau, \quad (68)$$

$$dp_{(2)} = \left[\bar{\psi}_{(2)}(i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi \bar{\psi}_{(1)} \right] d\tau, \quad (69)$$

$$d\bar{p}_{(1)} = \left[-(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi \psi_{(2)} \right] d\tau - i\gamma^0 d\psi_{(1)}, \quad (70)$$

$$d\bar{p}_{(2)} = \left[-(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)} \right] d\tau - i\gamma^0 d\psi_{(2)}. \quad (71)$$

The integrability conditions ($dH'_\alpha = 0$) imply that the variation of the constraints $H'_{(i)}$ and $\bar{H}'_{(i)}$ should be identically zero, that is

$$dH'_{(i)} = dp_{(i)} - i d\bar{\psi}_{(i)} \gamma^0 = 0, \quad (72)$$

$$d\bar{H}'_{(i)} = d\bar{p}_{(i)} = 0. \quad (73)$$

The following equations of motion:

From Eq. (64), we obtain

$$\dot{\varphi} = p_\varphi. \quad (74)$$

Substituting from Eqs. (68) and (69) into Eq. (72), we get

$$i\partial_0 \bar{\psi}_{(1)} \gamma^0 - \bar{\psi}_{(1)} (i\overleftarrow{\partial}_a \gamma^a + m_1) - g\varphi \bar{\psi}_{(2)} = 0, \quad (75)$$

$$i\partial_0 \bar{\psi}_{(2)} \gamma^0 - \bar{\psi}_{(2)} (i\overleftarrow{\partial}_a \gamma^a + m_2) - g\varphi \bar{\psi}_{(1)} = 0. \quad (76)$$

Substituting from Eqs. (70) and (71) into Eq. (73), we obtain

$$(i\gamma^\mu \partial_\mu - m_1)\psi_{(1)} - g\varphi \psi_{(2)} = 0, \quad (77)$$

$$(i\gamma^\mu \partial_\mu - m_2)\psi_{(2)} - g\varphi \psi_{(1)} = 0. \quad (78)$$

One notes that the integrability conditions are not identically zero, they are added to the set of equations of motion.

From Eqs. (67-69), we get the following equations of motion:

$$\dot{p}_\varphi = m^2 \varphi + \frac{1}{2} \lambda \varphi^2 + g(\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}), \quad (79)$$

$$\dot{p}_{(1)} = \bar{\psi}_{(1)} (i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi \bar{\psi}_{(2)}, \quad (80)$$

$$\dot{p}_{(2)} = \bar{\psi}_{(2)} (i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi \bar{\psi}_{(1)}. \quad (81)$$

Substituting from Eqs. (77) and (78) into (70) and (71), we get

$$\dot{\bar{p}}_{(i)} = 0, \quad i = 1, 2. \quad (82)$$

Differentiate Eq. (74) with respect to time, and making use of (79), we obtain

$$\ddot{\varphi} - m^2 \varphi - \frac{1}{2} \lambda \varphi^2 - g(\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}) = 0. \quad (83)$$

5 Hamilton-Jacobi quantization

In this section, we use the Hamilton-Jacobi quantization to obtain the path integral quantization of the scalar field coupled to two flavours of fermions through Yukawa couplings.

First, one must check whether the set of equations (64-71) is integrable or not. So, one has to consider the total variations of the constraints. In fact

$$dH'_{(i)} = dp_{(i)} - i d\bar{\psi}_{(i)} \gamma^0 = 0, \quad (84)$$

$$d\bar{H}'_{(i)} = d\bar{p}_{(i)} = 0. \quad (85)$$

The constraints (62) and (63), lead us to obtain $d\bar{\psi}_{(i)}$ and $d\psi_{(i)}$ in terms of dt

$$d\bar{\psi}_{(1)} i\gamma^0 = [\bar{\psi}_{(1)} (i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi \bar{\psi}_{(2)}] dt, \quad (86)$$

$$d\bar{\psi}_{(2)} i\gamma^0 = [\bar{\psi}_{(2)} (i\overleftarrow{\partial}_a \gamma^a + m_2) + g\varphi \bar{\psi}_{(1)}] dt, \quad (87)$$

$$i\gamma^0 d\psi_{(1)} = [-(i\gamma^a \partial_a - m_1)\psi_{(1)} + g\varphi \psi_{(2)}] dt, \quad (88)$$

and

$$i\gamma^0 d\psi_{(2)} = [-(i\gamma^a \partial_a - m_2)\psi_{(2)} + g\varphi \psi_{(1)}] dt. \quad (89)$$

We obtain that the set of equations (64-71) is integrable. Making use of (13), and (61-63), we can write the canonical action integral as

$$Z = \int d^4x \left[\frac{1}{2} (p_\varphi^2 + \partial_a \varphi \partial^a \varphi) - \frac{1}{2} m^2 \varphi^2 - \frac{1}{6} \lambda \varphi^3 + \bar{\psi}_{(i)} (i \gamma^\mu \partial_\mu - m_i) \psi_{(i)} - g \varphi (\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}) \right], \quad (90)$$

Now the path integral representation (19) is given by

$$\langle out | S | In \rangle = \int \prod_i^2 d\varphi dp_\varphi d\psi_{(i)} d\bar{\psi}_{(i)} \exp \left\{ i \left[\int d^4x \frac{1}{2} (p_\varphi^2 + \partial_a \varphi \partial^a \varphi) - \frac{1}{2} m^2 \varphi^2 - \frac{1}{6} \lambda \varphi^3 + \bar{\psi}_{(i)} (i \gamma^\mu \partial_\mu - m_i) \psi_{(i)} - g \varphi (\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}) \right] \right\}. \quad (91)$$

6 Conclusion

The scalar field coupled to two flavours of fermions through Yukawa couplings is discussed as constrained system [7] by using both Dirac's and Hamilton-Jacobi methods. In Dirac's method the total Hamiltonian composed by adding the constraints multiplied by Lagrange multipliers to the canonical Hamiltonian. In order to derive the equations of motion, one needs to redefine these unknown multipliers in an arbitrary way. However, in the Hamilton-Jacobi approach (or Güler's method) [2-8], there is no need to introduce Lagrange multipliers to the canonical Hamiltonian, then the Hamilton-Jacobi is simpler and more economical.

In Hamilton-Jacobi approach it is not necessary to distinguish between first-class and second-class constraints. Hamilton-Jacobi approach always in exact agreement with Dirac's method. Both the consistency conditions and the integrability conditions lead to the same constraints. Also the equations of motion in both approaches are the same.

Hamilton-Jacobi quantization is applied to the constrained field system with finite degrees of freedom by investigating the integrability conditions without using any gauge fixing condition.

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QUANTIZATION OF A MASSIVE SPIN ONE PARTICLE

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Abstract

The Hamilton-Jacobi approach is used to quantize a massive spin one particle, which is a constrained system. The equations of motion for this singular system are obtained as total differential equations in many variables. After writing the equations of motion in a compact notation for vectors, they become equations of the electromagnetic field and Maxwell's equations. The integrability conditions are investigated without using any gauge fixing condition. The canonical method is used to quantize the system. The path integral is obtained as an integration over the canonical phase space coordinates.

1 Introduction

The quantization of constrained Hamiltonian systems can be achieved by means of operator methods [1,2] or by path integral quantization [3-5].

The canonical method (or Güler's method) was developed the Hamilton-Jacobi formulation to investigate the quantization of constrained systems [6, 7]. The variational principle is the starting point of the Hamilton-Jacobi approach [8-11]. The treatment of constrained systems due to the Hamiltonian leads to obtain the equations of motion as total differential equations in many variables. the equations of motion are integrable when the corresponding system of partial differential equations is a Jacobi system. Güler has presented in Ref. [12] a classically treatment of constrained systems. Afterwards, the quantization of constrained system was investigated by Hamilton-Jacobi quantization as seen in Ref. [13] which is presented the Hamilton-Jacobi quantization of finite dimensional system with constraints. By comparison, Dirac's method with the canonical method (Hamilton-Jacobi approach) in treatment of constrained singular systems, one finds that both give the same results as seen in Refs. [14-18]. But the Hamilton-Jacobi approach is simpler than Dirac's method [19-23] because of: (i) the Hamilton-Jacobi formalism makes no difference between first- and second-class constraints (ii) the gauge fixing is not always an easy task in the case of using Dirac's method, but if the canonical method (Hamilton-Jacobi approach) [7,8] is used, it does not require a gauge-fixing term at all because the gauge variables are separated in the processes of constructing an integrable system of total differential equations.

The quantization of a massive spin one particle became the center of interest of physicists, especially after the pioneering work of Faddeev [3], who introduced the path integral quantization of singular theories that possesses first-class constraints in canonical gauge. Since first class constraints are generators of gauge transformations, which will lead to gauge freedom. In other words, the equations of motion are still degenerate and depending on the functional arbitrariness, one has to impose external gauge constraints for each first-class constraint. Senjanovic [5] generalized Faddeev's method for the quantization, and construction of the functional integral for theories with second-class constraints in canonical gauge. However,

based on the canonical method, the path integral formulation obtained in Refs. [23–25]. We investigated the path integral quantization of singular constrained field systems by using the Faddev’s method and canonical method in the case of first-class constraints as seen in Ref. [19], and by using the Senjanovic’s method and canonical method for the field system which contains first-class constraints and second-class constraints as seen in Ref. [23]. That leads to the same results and shows the simplicity for using the canonical path integral quantization for quantizing the systems.

The purpose of this paper is to use the path integral quantization of systems with finite degrees of freedom in order to include the electromagnetic field with constraints using the canonical method.

This paper is arranged as follows: canonical path integral quantization method is presented in Sec.2. the quantization method is applied to a massive spin one particle in Sec.3. The paper closes with a conclusion in Sec.4.

2 Canonical path integral quantization

In this section, we study the constrained systems by using the canonical method [6,7] and demonstrate the fact that the gauge fixing problem is solved naturally. The starting point of this method is to consider the lagrangian $L \equiv L(q_i, \dot{q}_i, \tau)$, $i = 1, 2, \dots, n$ with Hess matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, 2, \dots, n, \quad (1)$$

of rank $(n - r)$, $r < n$. Then r momenta are dependent. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \dots, n - r, \quad (2)$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \dots, n. \quad (3)$$

enable us to solve Eq. (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, p_a, \dot{q}_\mu; \tau) \equiv \omega_a. \quad (4)$$

Substituting Eq. (4), into Eq. (3), we get

$$p_\mu = \left. \frac{\partial L}{\partial \dot{q}_\mu} \right|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, p_a; t). \quad (5)$$

Relations (5) indicate the fact that the generalized momenta p_μ are not independent of p_a which is a natural result of the singular nature of the lagrangian.

The canonical Hamiltonian H_0 is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a \equiv \omega_a; \tau) + p_a \dot{q}_a + p_\mu \dot{q}_\mu \Big|_{p_\mu = -H_\mu}. \quad (6)$$

The set of Hamilton-Jacobi Partial Differential Equations is expressed as

$$H'_\alpha \left(\tau, q_\mu, q_a, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial \tau} \right) = 0, \quad \alpha = 0, n-p+1, \dots, n, \quad (7)$$

where

$$H'_\alpha = p_\alpha + H_\alpha. \quad (8)$$

The equations of motion are obtained as total differential equation in many variables as follows:

$$dq_r = \frac{\partial H'_\alpha}{\partial p_r} dt_\alpha, \quad r = 0, 1, \dots, n, \quad (9)$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad a = 1, \dots, n-p, \quad (10)$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \quad \alpha = 0, n-p+1, \dots, n, \quad (11)$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha. \quad (12)$$

where $Z = S(t_\alpha, q_a)$. The set of equations (9-12) are integrable if

$$dH'_\alpha = 0, \quad \alpha = 0, n-p+1, \dots, n. \quad (13)$$

If conditions (13) are not satisfied identically, one may consider them as new constraint and a gain test the integrability conditions ,then repeating

this procedure, a set of conditions may be obtained.

In this case of integrable system, the path integral representation may be written as [26-29].

$$\langle Out | S | In \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp \left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right]. \quad (14)$$

One should notice that the integral (14) is an integration over the canonical phase space coordinates q_a, p_a .

3 Path integral quantization of the massive spin one particle

The best example of a theory exhibiting gauge invariance is Maxwell electromagnetic. the following Lagrangian describe the theory

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3, \quad (15)$$

with the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and the Abelian gauge field A^μ which transforms under gauge transformation as $\delta A_\mu(x) = \partial_\mu \Lambda(x)$. where Λ is some arbitrary function of coordinates. Therefore, the field strength tensor $F_{\mu\nu}$ is a gauge invariant.

Schwinger [30] proved that the gauge invariant and mass can coexist. He showed, in the case of the gauge field is coupled to a conserved current J^μ and the theory is such that current correlator $\langle vac | J_\mu J_\nu | vac \rangle$ has a pole at $P^2 = 0^1$, the gauge bosons in the theory are invariably massive.

Therefore, the Lagrangian of a massive spin one particle is described as

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - A_\mu J^\mu, \quad \mu, \nu = 0, 1, 2, 3, \quad (16)$$

This Lagrangian function is singular, since the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}. \quad (17)$$

is three.

The momenta conjugated to the fields A^μ are defined as

$$\pi^\mu = \frac{\partial l}{\partial \dot{A}_\mu} = F^{\mu 0}. \quad (18)$$

Therefore

$$\pi^0 = 0 = -H. \quad (19)$$

The spatial components of π^μ are just the electric field:

$$\pi^i = F^{i0} = E^i, \quad i = 1, 2, 3. \quad (20)$$

Using (18), the velocities $\dot{A}_i$ can be expressed in terms of the momenta π_i as

$$\dot{A}_i = -\pi_i + \partial_i A_0. \quad (21)$$

Therefore, the Hamiltonian density is obtained as

$$H_0 = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi^i \pi_i + \pi^i \partial_i A_0 - \frac{1}{2} m^2 (A_0 A^0 + A_i A^i) + A_0 J^0 + A_i J^i. \quad (22)$$

The set of Hamilton-Jacobi partial differential equations reads

$$H'_0 = p_0 + H_0, \quad (23)$$

$$H' = \pi_0 + H = \pi_0 = 0. \quad (24)$$

Therefore, the total differential equations for the characteristic (9), (10) and (11) are

$$\begin{aligned} dA^i &= \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'}{\partial \pi_i} dA^0, \\ &= -(\pi^i + \partial_i A_0) dt, \end{aligned} \quad (25)$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'}{\partial A_i} dA^0, \\ &= (\partial_l F^{li} + m^2 A^i - J^i) dt, \quad l = 1, 2, 3, \end{aligned} \quad (26)$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'}{\partial A_0} dA^0, \\ &= (\partial_i \pi^i + m^2 A^0 - J^0) dt, \end{aligned} \quad (27)$$

$$\begin{aligned} dp^0 &= -\frac{\partial H'_0}{\partial t} dt - \frac{\partial H'}{\partial t} dA^0, \\ &= -(A^0 \frac{\partial J_0}{\partial t} + A^i \frac{\partial J_i}{\partial t}) dt. \end{aligned} \quad (28)$$

To check whether the set of Eqs. (25), (26), (27) and (28) is integrable or not, let us consider the variation of Eqs. (23) and (24). In fact

$$dH'_0 = F_1 dA^0, \quad (29)$$

where

$$F_1 = (\partial_i \pi^i + m^2 A_0 - J_0) \equiv 0. \quad (30)$$

Since (29) is not identically zero, we consider it as a new constraint F_1 , and one should consider the variation of F_1 too. The variation is simply zero, and it is equal to F_1 .

Imposing a gain the variation of F_1

$$dF_1 = d(\partial_i \pi^i + m^2 A_0 - J_0) = \partial_i d\pi^i + m^2 dA_0. \quad (31)$$

This equation is not identically zero since

$$\begin{aligned} \partial_i \pi^i &= \partial_i \left(-\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'}{\partial A_i} dA^0 \right), \\ &= \partial_i (m^2 A^i - J^i) dt. \end{aligned} \quad (32)$$

Therefore, we have another new constraint of the form

$$F_2 \equiv -\partial_i (J^i - m^2 A^i) = 0, \quad (33)$$

which represents the continuity equation. Finally, the variation of F_2 is identically zero since

$$dF_2 = -d[\partial_i (J^i - m^2 A^i)] \equiv 0. \quad (34)$$

Thus, no further constraints arise.

The set of Eqs. (25), (26), (27) and (28) is integrable and the equivalent partial differential equations of them are just ordinary differential equations which are take the form

$$\dot{A}^i = -\pi^i - \partial_i A_0, \quad (35)$$

$$\dot{\pi}^i = \partial_i F^{ti} + m^2 A^i - J^i, \quad (36)$$

$$\dot{\pi}^0 = \partial_i \pi^i + m^2 A^0 - J^0. \quad (37)$$

One may write Eqs. (35 - 37) in a compact notation for vectors as follows:

$$\frac{\partial \vec{A}}{\partial t} + \vec{E} = -\vec{\nabla} \phi, \quad (38)$$

$$\frac{\partial \vec{E}}{\partial t} = (\vec{\nabla} \times \vec{B}) + m^2 \phi - \vec{J}, \quad (39)$$

$$\vec{\nabla} \cdot \vec{E} = -m^2 \phi + \rho, \quad (40)$$

where $\partial_i F^{ti} = (\vec{\nabla} \times \vec{B})_i$.

Taking the curl of Eq. (38) yields

$$\frac{\partial \vec{B}}{\partial t} + (\vec{\nabla} \times \vec{E}) = 0, \quad (41)$$

where $\vec{B} = (\vec{\nabla} \times \vec{A})$, which gives

$$\vec{\nabla} \cdot \vec{B} = 0. \quad (42)$$

Equations (39) and (40) are equations for the electromagnetic field, whereas Eqs. (41) and (42) are Maxwell's equations.

Eqs. (12) and (22 - 24) lead us to the canonical action integral as

$$\begin{aligned} Z = \int d^4x \Big(& -\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi^i \pi_i + \pi_i \partial_i A_0 + \pi_i \dot{A}^i \\ & + \frac{1}{2} m^2 (A_0 A^0 + A_i A^i) - A_0 J^0 - A_i J^i \Big). \end{aligned} \quad (43)$$

Making use of Eqs. (14) and (43), we obtain the path integral as

$$\langle out|S|In\rangle = \int \prod DA^i D\pi^i \exp \left[i \left\{ \int d^4x \left(\frac{1}{2} \pi^2 + \pi_i (\partial_i A_0 + \dot{A}^i) \right. \right. \right. \\ \left. \left. \left. + \frac{1}{2} m^2 (A_0 A^0 + A_i A^i) - \frac{1}{4} F^{ij} F_{ij} - A_0 J^0 - A_i J^i \right) \right\} \right]. \quad (44)$$

Integration over π_i gives

$$\langle out|S|In\rangle = N \int \prod DA^i \exp \left[i \left\{ \int d^4x \left(\frac{1}{2} (\partial_i A^0 + \dot{A}^i)^2 - \frac{1}{4} F^{ij} F_{ij} \right. \right. \right. \\ \left. \left. \left. + \frac{1}{2} m^2 (A_0 A^0 + A_i A^i) - A_0 J^0 - A_i J^i \right) \right\} \right], \\ = N \int \prod DA^i \exp \left[i \left\{ \int d^4x \left(\frac{1}{2} \dot{A}^2 - \frac{1}{4} F^{ij} F_{ij} + J \cdot A \right. \right. \right. \\ \left. \left. \left. + \frac{1}{2} m^2 (A_0 A^0 + A_i A^i) \right. \right. \right. \\ \left. \left. \left. - \frac{1}{2} A^0 (2 \nabla \cdot \dot{A} + 2 J_0 + \nabla^2 A_0) \right) \right\} \right], \quad (45)$$

where N is a constant.

Equation (45) can be written as

$$\langle out|S|In\rangle = N \int \prod DA^i \exp \left[i \left\{ \int d^4x \left(\frac{1}{2} \dot{A}^2 - \frac{1}{4} F^{ij} F_{ij} + J \cdot A \right. \right. \right. \\ \left. \left. \left. + \frac{1}{2} m^2 (\phi^2 + A_i A^i) - \frac{1}{2} \phi (2 \nabla \cdot \dot{A} + 2 \rho + \nabla^2 \phi) \right) \right\} \right], \quad (46)$$

Equations (38) and (40) lead us to obtain

$$\nabla \cdot \dot{A} = -\nabla^2 \phi + m^2 \phi - \rho. \quad (47)$$

Inserting Eq. (47) in the integral (46) gives

$$\langle out|S|In\rangle = N \int \prod DA^i \exp \left[i \left\{ \int d^4x \left(\frac{1}{2} \dot{A}^2 - \frac{1}{4} F^{ij} F_{ij} \right. \right. \right. \\ \left. \left. \left. + J \cdot A + \frac{1}{2} \phi \nabla^2 \phi - \frac{1}{2} m^2 (\phi^2 - A_i A^i) \right) \right\} \right]. \quad (48)$$

4 Conclusion

We have obtained the path integral quantization of a massive spin one particle, which is a constrained field system, by using the canonical method. The equations of motion of this system are derived.

One should notice that the term $\frac{1}{2} \dot{A}^2 - \frac{1}{4} F^{ij} F_{ij} + J \cdot A$ leads to a solution for A , in which

$$A = \int G \eta d^3x. \quad (49)$$

Thus G has an inverse. Hence, the path integral (49) has no singular nature.

As was demonstrated by this example, gauge fixing is not necessary to obtain the path integral formulation for field theories if the canonical formulation is used. Since this system is integrable, H'_0 and H' can be interpreted as infinitesimal generators of canonical transformations given by parameters t and A^0 , respectively. In this case the path integral is obtained as an integration over the canonical phase-space coordinates A^i, π^i . Faddeev, Popov and Senjanovic treatments [3-5] need gauge-fixing conditions to obtain the path integral over the canonical variable.

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PROPERTIES SUPERSOLUBLE GROUPS

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Abstract

A metabelian group is a group G that possesses a normal subgroup N such that N and G/N are both abelian. Equivalently, G is metabelian if and only if the commutator subgroup $[G, G]$ is abelian. Equivalently again, G is metabelian if and only if it is solvable of length at most 2

Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B .

Then $G/F(G)$ is supersoluble.

Keywords: mutually permutable product- totally permutable, supersoluble subgroups

1. Introduction.

It is known that a group which is the product of two super soluble groups is not necessarily super soluble, even if the two factors are normal subgroups of the group. Baer proved in [3] that if a group G is the product of two normal supersoluble groups and G' is nilpotent, then G is super soluble. The search for generalizations of Baer's result has been a fruitful topic of investigation recently (see [3,4]).

subgroups A and B such that $A \cap B = 1$, then the product is in fact totally permutable [6, Proposition 3.5], and therefore G is supersoluble. Our main Theorem is a generalization of this last property.

Theorem 1.

Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B . If $\text{Core}_G(A \cap B) = 1$, then G is supersoluble.

Lemma 1[4, Theorem 2]. If $G=AB$ is the mutually permutable product of the supersoluble subgroups A and B , then G is soluble.

Lemma 2. Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B . Then, either G is supersoluble or $N_A < G$ and $N_B < G$ for every minimal normal subgroup N of G .

Proof. Assume that G is not supersoluble. Then both A and B are proper subgroups of G . Let N be a minimal normal subgroup of G and for contradiction assume that $N_A = G$. Then, as N is abelian, $N \cap A$ is a normal subgroup of

$\langle N, A \rangle = G$. Since N is a minimal normal subgroup of G and $A < G$, we have that $N \cap A = 1$ and consequently A is a maximal subgroup of G . Clearly we can also assume that B is not contained in A . It is not difficult to argue that we can choose an element b of $B \setminus A$ such that $bq \in A$ for some prime q . Since the product $G=AB$ is mutually permutable, $A \langle b \rangle$ is a subgroup of G and the maximality of A implies that $G = A \langle b \rangle$. We now take orders to reach our final contradiction:

$|A||N| = |G| = |A| |\langle b \rangle| |A \cap \langle b \rangle| = q|A|$. Consequently we have that $|N| = q$ and then G is supersoluble, a contradiction.

Lemma 3. Let $G=AB$ be the mutually permutable product of the subgroups A and B and let N be any minimal normal subgroup of G . Then either

$$N \cap A = N \cap B = 1 \text{ or } N = (N \cap A)(N \cap B).$$

Proof. Let N be a minimal normal subgroup of G . Clearly $A(N \cap B)$ and $(N \cap A)B$ are both subgroups of G . Note that A normalizes $N \cap (A(N \cap B)) = (N \cap A)(N \cap B)$ and B normalizes $N \cap ((A \cap N)B) = (N \cap A)(N \cap B)$. Therefore $(N \cap A)(N \cap B)$ is a normal subgroup of G and the minimality of N yields the result.

Lemma 4. Let G be a group, and N a minimal normal subgroup of G such that $|N|=pn$, where p is a prime and $n>1$. Denote $C=CG(N)$ and assume that G/C is supersoluble. Then, if Q/C is a subgroup of G/C containing $Op'(G/C)$, we have that Q is normal in G and $N=\prod_{i=1}^t N_i$, where N_i are non-cyclic minimal normal subgroups of NQ for $i=1, \dots, t$.

Proof. Since by [8, Lemma A.13.6], we have that $Op(G/C)=1$ and the commutator subgroup $(G/C)'$ of G/C is nilpotent because G/C is supersoluble, it follows that $(G/C)'$ is a p' -group. Therefore $(G/C)'$ is contained in $Op'(G/C)$ and thus $Op'(G/C)$ is a Hall p' -subgroup of G/C . Consequently, Q/C is a normal subgroup of G/C and hence Q is normal in G . Consider now N as a G -module over $GF(p)$ by conjugation. Then, by Clifford's Theorem [8, Theorem B.7.3], N viewed as a Q -module is a direct sum $N=\prod_{i=1}^t N_i$, where N_i are irreducible Q -modules for $i=1, \dots, t$. Suppose that there exists $i \in \{1, \dots, t\}$ such that $|N_i|=p$. Then clearly $|N_j|=p$ for all j . Therefore $Q/CQ(N_i)$ is abelian of exponent dividing $p-1$, and the same is true for Q/C . In particular, $Q/C=Op'(G/C)$ is a Hall p' -subgroup of G/C . Since N is not cyclic, it follows that $Q=G$ and thus p divides $|G/C|$. Hence p is the largest prime dividing $|G/C|$. From the supersolubility of G/C , we obtain that $1=Op(G/C)$ is a Sylow subgroup of G/C , a contradiction. Consequently, N_i is a non-cyclic minimal normal subgroup of NQ for all $i \in \{1, \dots, t\}$, as we wanted to prove.

Proof of Theorem 1. Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B , with $CoreG(A \cap B)=1$, and suppose that G has been chosen minimal such that its supersoluble residual GU is non-trivial. Let N be a minimal normal subgroup of G contained in GU . Note that N is an elementary abelian p -group for some prime p . Applying Lemma 2, we have that both NA and NB are proper subgroups of G . Moreover, using Lemma 3, we have that either $N=(N \cap A)(N \cap B)$ or $N \cap A=N \cap B=1$. Assume first that $N=(N \cap A)(N \cap B)$.

(i) If $N \cap A=1$, then N is cyclic. Assume that $N \cap A=1$. It follows that N is contained in B . Let N_0 be a non-trivial cyclic subgroup of N . Since AN_0 is a subgroup of G , we have that $N_0=AN_0 \cap N$ is a normal subgroup of AN_0 . Hence every cyclic subgroup of N is normalised by A . Now let N_1 be a minimal normal subgroup of B contained in N . Since B is supersoluble, it follows

That N_1 is cyclic and thus normalised by A . Hence N_1 is a normal subgroup of G . The minimality of N implies that $N=N_1$ and consequently N is cyclic.

(ii) $N \cap A=1$ and $N \cap B=1$. On the contrary, assume that $N \cap A=1$. From (i), we know that N is cyclic. Moreover, N is contained in B . Hence $AN \cap B=(A \cap B)N$. Let $L=CoreG(A \cap B)N$. Clearly, N is contained in L and $L=L \cap ((A \cap B)N)=(L \cap A \cap B)N$. It is clear that $G/L=(AL/L)(BL/L)$ is a mutually permutable product of AL/L and BL/L such that $CoreG/L((AL/L) \cap (BL/L))=1$. By the minimality of G , it follows that G/L is supersoluble. On the other hand, since N is cyclic, we have that $G/CG(N)$ is abelian. Hence $G/CL(N)$ is supersoluble and $GUCL(N)=C$. Note that $C=N \times (C \cap A \cap B)$.

Therefore $C \cap A \cap B$ contains a Hall p' -subgroup of C . Since $\text{Core}_G(A \cap B) = 1$ and $\text{Op}'(C)$ is a normal subgroup of G contained in $C \cap A \cap B$, we have that $\text{Op}'(C) = 1$. Moreover, $C' = (C \cap A \cap B)'$ is a normal subgroup of G contained in $A \cap B$. Consequently, $C' = 1$ and C is an abelian p -group. In particular, GU is abelian and thus GU is complemented in G by a supersoluble normalizer D which is also a supersoluble projector of G , by [8, Theorems V.4.2 and V.5.18]. Since N is cyclic, we know that N is central with respect to the saturated formation of all supersoluble groups. By [8, Theorem V.3.2.e], D covers N and thus N is contained in D . It follows $ND \cap GU = 1$, a contradiction.

(iii) Either $N = N \cap A$ or $N = N \cap B$. If we have $N = N \cap A = N \cap B$, then N is contained in $A \cap B$, contradicting the fact that $\text{Core}_G(A \cap B) = 1$. We may assume without loss of generality that $N \cap A = N$.

(iv) AN and BN are both supersoluble. Since $N = (N \cap A)(N \cap B)$ and $N = N \cap A$, it follows that $N \cap B$ is not contained in $N \cap A$. Let n be any element of $N \cap B$ such that $n \notin N \cap A$, and write $N_0 = \langle n \rangle$. Note that AN_0 is a subgroup of G , and $AN_0 \cap N = (N \cap A)N_0$. Therefore $N_0(N \cap A)$ is a normal subgroup of AN_0 , and consequently A normalizes $(A \cap N)N_0$. This yields that $A/CA(N/N \cap A)$ acts as a power automorphism group on $N/N \cap A$. This means that AN is supersoluble. If $N \cap B = N$, then $BN = B$ is supersoluble. On the contrary, if $N \cap B \neq N$, we can argue as above and we obtain that BN is supersoluble. Consequently, $\text{ACG}(N)/\text{CG}(N)$ and $\text{BCG}(N)/\text{CG}(N)$ are both abelian groups of exponent dividing $p-1$. But then $G/\text{CG}(N) = (\text{ACG}(N)/\text{CG}(N))(\text{BCG}(N)/\text{CG}(N))$ is a π -group for some set of primes π such that if $q \in \pi$, then q divides $p-1$.

(v) Let B_0 be a Hall π -subgroup of B . Then $AB_0 \cap N = A \cap N$.

This follows just by observing that $AB_0 \cap N$ is contained in each Hall π' -subgroup of AB_0 and every Hall π' -subgroup of A is a Hall π' -subgroup of AB_0 . Note that $|G/\text{CG}(N)|$ is a π -number and AB_0 contains a Hall π -subgroup of G . Therefore $G = (AB_0)\text{CG}(N)$. But then $A \cap N$ is a normal subgroup of G . The minimality of G yields either $A \cap N = 1$ or $A \cap N = N$. This contradicts our assumption $1 = N \cap A = N$, and so we cannot have $N = (A \cap N)(B \cap N)$. Thus, by Lemma 3, we may assume $N \cap A = N \cap B = 1$. Let $M = \text{Core}_G(AN \cap BN)$. Then $N \cap M = N$ and G/M is supersoluble by the minimality of G . Again, we reach a contradiction after several steps.

(vi) $M = N$. Suppose that $M = N$. Since G/M is supersoluble, we know that N cannot be cyclic. Let us write $C = \text{CG}(N)$, and consider the quotient group G/C . It is clear that G/C is supersoluble. Let $Q/C = \text{Op}(G/C)$. Since $\text{Op}(G/C) = 1$ and $(G/C)'$ is nilpotent, it follows that Q/C is a normal Hall p' -subgroup of G/C . Let $B_{p'}$ be a Hall p' -subgroup of B . Since $|N|$ divides $|B : A \cap B|$, we have that $(A \cap B)B_{p'}$ is a proper subgroup of B . Let T be a maximal subgroup of B containing $(A \cap B)B_{p'}$. Then AT is a maximal subgroup of G and $|G : AT| = p = |B : T|$. If N is not contained in AT , we have $G = (AT)N$

and $AT \cap N = 1$. Then $|N| = p$, a contradiction. Therefore N is contained in AT . In particular, the family $S = \{X : X \text{ is a proper subgroup of } B, (A \cap B)Bp'X \text{ and } NAX\}$ is non-empty. Let R be an element of S of minimal order. Observe that AR has p -power index in G and thus ARC/C contains $Op'(G/C)$. Regarding N as a AR -module over $GF(p)$, we know, by Lemma 4, that N is a direct sum $N = \prod_{i=1}^t N_i$, where N_i is an irreducible AR -module whose dimension is greater than 1, for all $i \in \{1, \dots, t\}$. Assume that $(A \cap B)Bp' = R$. Then $AR = ABp'$ and thus N is contained in A , a contradiction. Therefore $ABp' \cap B = (A \cap B)Bp'$ is a proper subgroup of R . Let S be a maximal subgroup of R containing $(A \cap B)Bp'$. From the minimality of R , we know that N is not contained in AS . Consequently, there exists some $i \in \{1, \dots, t\}$ such that N_i is not contained in AS , which is a maximal subgroup of AR . Hence $AR = (AS)N_i$. Since N_i is a minimal normal subgroup of AR , it follows that $AS \cap N_i = 1$ and $|N_i| = |AR : AS| = |R : S| = p$, a contradiction.

(vii) M is an elementary abelian p -group. Note that $M = N(M \cap A) = N(M \cap B)$ and $|M \cap A| = |M \cap B| = |M|/|N|$. Moreover, $A(M \cap B)$ is a subgroup of G such that $A(M \cap B) \cap M = (M \cap A)(M \cap B)$. Hence $(M \cap A)(M \cap B)$ is also a subgroup of G . If $M \cap A = M \cap B$, then $M \cap A$ is a normal subgroup of G contained in $A \cap B$. This implies that $M \cap A = 1$ and consequently $M = N$, a contradiction. It yields that $M \cap A \neq M \cap B$. Next we see that $(M \cap A)(M \cap B)$ is a normal subgroup of G . Since $(M \cap A)(M \cap B) = M \cap A(M \cap B)$, we have that A normalizes $(M \cap A)(M \cap B)$. Similarly, B normalises

$(M \cap A)(M \cap B)$ since $(M \cap A)(M \cap B) = M \cap B(M \cap A)$. This implies normality of $(M \cap A)(M \cap B)$ in G . Let $X = (M \cap A)(M \cap B)$. Since we cannot have $M \cap A = M \cap B$, $M \cap A$ must be strictly contained in X . Thus $X = X \cap M = (X \cap N)(M \cap A) > M \cap A$ gives us $X \cap N = 1$. But then $X \cap N = N$, giving NX . Suppose that Q is a Hall p' -subgroup of $M \cap B$. Then QA is a subgroup and so $QA \cap M = Q(M \cap A)$ is also a subgroup which contains Q . Hence, as $|M : M \cap A| = p^k$ for some k , we have that $QM \cap A \cap B$. Thus $QB \cap MM \cap A \cap B$ and similarly $QA \cap MM \cap A \cap B$. Consequently, QM is contained in $M \cap A \cap B$. Since $QM = Op(M)$, it follows that $Op(M)$ is a normal subgroup of G contained in $A \cap B$. Hence $Op(M) = 1$, a contradiction, and consequently $Q = 1$ and M is a p -group. Hence N is contained in $Z(M)$ and $M = N \times (M \cap A) = N \times (M \cap B)$. Thus $\phi(M) = \phi(M \cap A) = \phi(M \cap B)$ is a normal subgroup of G contained in $A \cap B$. This implies that $\phi(M) = 1$ and M is an elementary abelian p -group, as claimed.

(viii) Final contradiction. We have from the previous steps that $M \cap A$ is not contained in $M \cap B$ and that $M \cap B$ is not contained in $M \cap A$ because otherwise, since $|M \cap A| = |M \cap B|$, it follows that $M \cap A = M \cap B$ is a normal subgroup of G contained in $A \cap B$. This would imply $M \cap A = M \cap B = 1$, and $M = (M \cap A)N = N$. This fact contradicts step (vi).

Let x be an element of $M \cap B$ such that $x \notin M \cap A$. Then $A \langle x \rangle$ is a subgroup of G , and so is $M_0 = A \langle x \rangle \cap M = (A \cap M) \langle x \rangle$. Therefore M_0 is an A -invariant subgroup of G . In particular, since $M = (M \cap A)(M \cap B)$, we have that every subgroup of $M/M \cap A$ is A -invariant; that is, $A/CA(M/M \cap A)$ acts as a group of power automorphisms on $M/M \cap A$. It is clear that $M/M \cap A$ is A -isomorphic to N .

Consequently, $A/CA(N)$ acts as a group of power automorphisms on N . This implies that A normalises each subgroup of N . Analogously, B normalises each subgroup of N . It follows that N is a cyclic group. We argue as in step (ii) above to reach a final contradiction. We have that G/M is supersoluble and M is abelian. Therefore GU and thus GU is abelian and complemented in G by a supersoluble normaliser, D say, by [8, Theorem V.5.18]. Since N is cyclic, we know that D covers N and thus $NGU \cap D = 1$, a contradiction. Proof of Theorem 2.

Let $M=GU$ denote the supersoluble residual of G . Theorem 1 yields that $G/\text{Core}G(A \cap B)$ is supersoluble. Therefore M is contained in $\text{Core}G(A \cap B)$. In particular, M is supersoluble. Let $F(M)$ be the Fitting subgroup of M . Since A and B are supersoluble, we have that $[M, A]F(A) \cap MF(M)$ and $[M, B]F(B) \cap MF(M)$. Consequently, $[M, G]$ is contained in $F(M)$. Note now that the chief factors of G between $F(M)$ and M are cyclic, and recall that G/M is supersoluble. Therefore we have that $G/F(M)$ is supersoluble. This implies that $M=F(M)$ and thus M is nilpotent. Consequently, $G/F(M)$ is supersoluble. We now show that $G/F(M)$ is metabelian. We prove first that A' and B' both centralise every chief factor of G . Let H/K be a chief factor of G . If H/K is cyclic, then as G' centralizes H/K , so do A' and B' . Hence we may assume that H/K is a non-cyclic p -chief factor of G for some prime p . Note that we may assume that H is contained in M because G/M is supersoluble and H/K is non-cyclic. To simplify notation, we can consider $K=1$. Since $F(G)$ centralizes H [8, Theorem A.13.8.b], $G/CG(H)$ is supersoluble. Let Ap' be a Hall p' -subgroup of A . By Maschke's theorem [8, Theorem A.11.5], H is a completely reducible Ap' -module and HAp' is supersoluble because H is contained in A . Therefore $Ap'/CAp'(H)$ is abelian of exponent dividing $p-1$. This implies that the primes involved in $|A/CA(H)|$ can only be p or divisors of $p-1$. The same is true for $|B/CB(H)|$. This implies that if p divides $|G/CG(H)|$, then p is the largest prime dividing $|G/CG(H)|$. But since $\text{Op}(G/CG(H))=1$ and $G/CG(H)$ is supersoluble, it follows that $G/CG(H)$ must be a p' -group. Consider H as A -module over $GF(p)$. Since $ACG(H)/CG(H)$ is a p' -group, we have that H is a completely reducible A -module and every irreducible A -submodule of H is cyclic. Consequently A' centralizes H , and the same is true for B' . Let now U/V be a chief factor of G . Then $G/CG(U/V)$ is the product of the abelian subgroups $ACG(U/V)/CG(U/V)$ and $BCG(U/V)/CG(U/V)$. By Itô's theorem [9], we have that $G/CG(U/V)$ is metabelian. Since $F(G)$ is the intersection of the centralisers of all chief factors (again by [8, Theorem A.13.8.b]), we can conclude that $G/F(G)$ is metabelian. 3. Final remarks Finally, Theorem 1 enables us to give succinct proofs of earlier results on mutually permutable products. is a cyclic group as well. This implies that G is supersoluble, a contradiction.

Main Theorem. Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B . Then $G/F(G)$ is supersoluble and metabelian.

Proof: by Theorem 1

Notice: This last theorem cannot be improved easily, as the following example shows.

Example. Let S_3 be the symmetric group of degree 3, given by

$S_3 = \langle \alpha, \beta : \alpha^2 = \beta^3 = 1, \beta\alpha = \beta^2\alpha \rangle$ and let T_7 be the non-abelian group of order 73 and exponent 7. Write $T_7 = \langle a, b \rangle$ with $a^7 = b^7 = [a, b]^7 = 1$ and let $c = [a, b]$. We have that S_3 acts on T_7 in the following way: $\alpha a = b, \alpha b = a, \alpha c = c^{-1}, \beta a = a^2, \beta b = b^4, \beta c = c$. Thus we can consider the semidirect product $G = [T_7] S_3$. Take now the subgroups

$A = T_7 \langle \beta \rangle$ and $B = T_7 \langle \alpha \rangle$ of G . Clearly both A and B are supersoluble, and it is easy to check that $G = AB$ is the mutually permutable product of A and B . Finally, we show that Theorem 1 provides elementary proofs for the results of Asaad and Shaalan about mutually permutable products.2.

Main results The following four lemmas are needed to prove Theorem 1.

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